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P-*m*-Mitotic Sets and Arithmetical Hierarchy

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Abstract

Let $\{0,1\}^*$ be the set of all finite strings of elements from $\{0,1\}$, and let **P** be the class of problems recognized by deterministic Turing machines, which run in polynomial time (a problem is simply a subset of $\{0,1\}^*$). This article defines the class $\widehat{\mathbf{P}}$ and shows that $\widehat{\mathbf{P}}$ is isomorphic to the class **P**.
Based on the notions of *T*-mitoticity and *T*-autoreducibility, K.Ambos-Spies introduced the notions of *P*-*m*-mitoticity and *P*-*m*-autoreducibility. The notions of $\widehat{\mathrm{P}}$-*m*-mitoticity and $\widehat{\mathrm{P}}$-*m*-autoreducibility are introduced by analogy with the mentioned notions.
The article proves that the index sets $\{z \mid W_z$ is $\widehat{\mathrm{P}}$-*m*-mitotic$\}$ and $\{z \mid W_z$ is $\widehat{\mathrm{P}}$-*m*-autoreducible$\}$ are Σ_3-complete.
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1. Introduction

Information about the basic concepts of computability theory used in this article, in particular the Turing machine (TM), the numbering of computably enumerable sets $\{W_i\}_{i\in\omega}$ and the arithmetical hierarchy, can be found in Rogers [1], Soare [2].

The two definitions of polynomial time reducibility given by Karp [3] and Cook [4] are just time-bounded versions of many-one reducibility $(\leq_m)$ and Turing reducibility $(\leq_T)$.

Among other works devoted to the research of time-bounded computations and used in this article, we note the works of Ladner [5], Ambos-Spies [6], Hopcroft, Ullman [7](1979), Sipser [8], Arora, Barak [9], Terwijn [10].

Notation. We fix the alphabet $\Lambda = \{0,1\}$.

Given a set Y, the set of all finite strings of elements from Y is denoted by Y^*.

A Turing machine T (deterministic or nondeterministic) runs in polynomial time if there is a polynomial function q such that for every input of length n, any computation sequence of T halts in $q(n)$ or fewer moves.

It is an intuitively appealing notion that **P** is the class of problems that can be solved efficiently.

In this article, we consider the class $\widehat{\mathbf{P}}$ (see Definition 2). Proposition 1 (below Definition 3) shows that the classes **P** and $\widehat{\mathbf{P}}$ are isomorphic (i.e., there is an isomorphic mapping from **P** to $\widehat{\mathbf{P}}$ and vice versa, there is an isomorphic mapping from $\widehat{\mathbf{P}}$ to **P**; with respect to the relations in question) (see the definition of isomorphic mapping in Definition 1).

An oracle Turing machine runs in polynomial time if there exists a polynomial function q such that for *every* input of length n and *any* oracle set X, the machine halts within $q(n)$ steps (see Ladner [5], p.156).

Note that the definitions of R. Ladner [11] and other authors are based on the concept of a multitape Turing machine.

Based on the notions of *T*-mitoticity and *T*-autoreducibility, Ambos-Spies [6] introduced the notions of *P-m*-mitoticity and *P-m*-autoreducibility. By analogy with the mentioned notions we introduce the notions of $\widehat{\mathrm{P}}$-m-mitoticity and $\widehat{\mathrm{P}}$-m-autoreducibility (see Definitions 15,16) and also give the definitions of index sets *M*(*P-m*)=$\{z \mid W_z$ is $\widehat{\mathrm{P}}$-m-mitotic$\}$ and *A(P-m)* = $\{ z \mid W_z$ is $\widehat{\mathrm{P}}$-T-autoreducible$\}$.

This article studies the location of index sets $\{z \mid W_z$ is $\widehat{\mathrm{P}}$-m-mitotic$\}$ and $\{z \mid W_z$ is $\widehat{\mathrm{P}}$-m-autoreducible$\}$ in the arithmetical hierarchy.

2. Preliminaries

Notation. Let ω be the set of all nonnegative integers.

We will denote the Λ^* elements by lowercase Greek letters $\sigma, \tau, \ldots$

Let us denote that σ^τ denote the *concatenation* of string σ followed by τ.

Let < be the natural order on Λ^* $(\lambda < 0 < 1 < 00 < 01 < \cdots)$, where λ represents the empty string.

We will denote the subsets ofΛ^* by uppercase Greek letters $\Xi, \Theta, \cdots,$ as well as by the Latin letter P with subscripts (P_i).

If $\sigma \in \Lambda^*$, then $|\sigma|$ denote the length of σ.

If $\Xi \subseteq \Lambda^*$, then

$$\Xi(\sigma) = \begin{cases} 1, & \text{if } \sigma \in \Xi \\ 0, & \text{if } \sigma \notin \Xi. \end{cases}$$

If $A \subseteq \omega$, then $A(x) = \chi_A(x)$(where χ_A is a characteristic function of a set A).

Define the mappings h_0, h_1 as follows:

Let h_0 be a 1-1 mapping from ω onto Λ^*, $h_0(0) = \lambda$, $h_0(n+1) = (n+2)$-nd string according to the order of strings on Λ^*.

Let h_1 be a 1-1 mapping from Λ^*onto ω.

$h_1(\lambda) = 0$;

$h_1(n+1$ string according to the order of strings on $\Lambda^*) = n$ (In fact, $h_1 = h_0^{-1}$).

Definition 1. (*i*) Let two sets $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ be given. Let there be defined any sort of relations between the elements of each of these sets.

If it is possible to place the two sets into one-to-one correspondence so that the mapping preserves the relations; that is, if with every element a of $\mathfrak{M}$ there can be associated an element b of $\widetilde{\mathfrak{M}}$ in a biunique manner so that the relations existing between any elements $a, b, \cdots$ of $\mathfrak{M}$ also exist between the associated elements $\bar{a}, \bar{b}, \cdots$ and vice versa, then the two sets are called *isomorphic* (with respect to the relations in question), and we write $\mathfrak{M} \cong \widetilde{\mathfrak{M}}$. The mapping itself is called an *isomorphism* (see Waerden [11], pp. 25-26).

(*ii*) If in two sets $\mathfrak{M}$ and $\mathfrak{N}$ certain relations are defined (such as $a < b$ or $ab = c$) and if to each element a of $\mathfrak{M}$ an image $\bar{a} = \varphi a$ is assigned in such a manner that all relations between the elements of $\mathfrak{M}$ also hold for the images (so that, for example, $a < b$ implies $\bar{a} < \bar{b}$ in the cases of the relation <), then φ is called a *homomorphic mapping* or *homomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ (see Waerden [11], p. 28).

Remark. It can be proved that the mapping $h_1: \Lambda^* \to \omega$ is an isomorphism.

It is known that there exist effective enumerations of the sets $P_0, P_1, \ldots$ and oracle Turing machines $\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots$, where P_i denotes the set recognized by the Turing machine (also denoted by P_i), which runs in polynomial time, and $\boldsymbol{M}_i$ denotes the oracle Turing machine, which runs in polynomial time. $\boldsymbol{M}_i(A)$ denotes the set recognized by $\boldsymbol{M}_i$ with oracle A (see Ladner [5], p.157).

Notation. For a given function f, $f \upharpoonright x$ denotes the restriction of f to arguments $y<x$, and $A \upharpoonright x$ denotes $\chi_A \upharpoonright x$.

(Note that any string $\sigma \in \Lambda^*$ can be considered as a partial function from ω into Λ.)

Let $h_0(A) = \{\tau | (\exists x)\,[h_0(x) = \tau \;\&\; x \in A]\}, h_1(\Xi) = \{x | (\exists \tau)[h_1(\tau) = x \;\&\; \tau \in \Xi]\}$.

Let $\sigma \in \Lambda^*$. By σ' we denote a string γ such that $h_0(\gamma) = h_0(\sigma) + 1$.

Let $\hat{h}$ be a computable function from ω onto ω^2.

Let Q_e be the Turing program with code number e (also called *index e*) in the standard listing (of programs), and let φ_e be the partial function computed by Q_e (see Soare [2], p.14).

We write $\varphi_{e,s}(x) = y$ if $x, y, e < s$ and y is the output of $\varphi_e(x)$ in $< s$ steps of the Turing program Q_e. If such a y exists, we say $\varphi_{e,s}(x)$ *converges*, which we write as $\varphi_{e,s}(x)\downarrow$, and $\varphi_{e,s}(x)\uparrow$ otherwise. Similarly, we write $\varphi_e(x)\downarrow$ if $\varphi_{e,s}(x)\downarrow$ for some s, and we write $\varphi_e(x)\downarrow= y$ if $\varphi_e(x)\downarrow= y$ and $\varphi_e(x) = y$ and similarly for $\varphi_{e,s}(x)\downarrow= y$ (see Soare [2], pp.16-17).

$W_e = \text{dom}\,\varphi_e$.

Based on the available numbering of computably enumerable (c.e.) sets $\{W_i\}_{i\in\omega}$, the available numbering of computable operators $\{\boldsymbol{\Phi}_i\}_{i\in\omega}$, and the available enumeration of polynomials $\{q_i\}_{i\in\omega}$, we define for an arbitrary i (proceeding from the fact that $\hat{h}(i) = (i_0, i_1)$)

1) the *set* $\widehat{P}_i$ as follows: $(\forall x)(\forall s \geq q_{i_1}(x))\left[\widehat{P}_{i,s}(x) = W_{i_0,q_{i_1}(x)}(x)\right]$,

   it is obvious that $(\forall x)(\forall s \geq q_{i_1}(x))\left[\widehat{P}_{i,q_{i_1}(x)}(x) = \widehat{P}_{i,s}(x) =_{dfn} \widehat{P}_i(x)\right]$;

2) the *oracle Turing machine* $\widehat{\boldsymbol{M}}_i$ as follows:
$(\forall x)\left(\forall s \geq q_{i_1}(x)\right)(\forall\sigma)\left[\widehat{\boldsymbol{M}}_{i,s}(\sigma)(x) = \boldsymbol{\Phi}_{i_0,q_{i_1}(x)}(\sigma)(x)\right]$,

it is obvious that $(\forall x)\left(\forall s \geq q_{i_1}(x)\right)(\forall\sigma)\left[\widehat{\boldsymbol{M}}_{i,q_{i_1}(x)}(\sigma)(x) = \widehat{\boldsymbol{M}}_{i,s}(\sigma)(x) =_{dfn} \widehat{\boldsymbol{M}}_i(\sigma)(x)\right]$.

Definition 2. $\widehat{\mathbf{P}} = \{\widehat{P}_i \mid i \in \omega\}$ (note, that $\mathbf{P} = \{P_i \mid i \in \omega\}$).

Based on the above and similar statements , which are also presented, for example, by Hopcroft, Ullman [7], Sipser [8], Arora, Barak [9],Terwijn [10], the following conclusion is presented in [9]:

All low-level choices (number of tapes, alphabet size, etc..) in the definition of Turing machines are immaterial, as they will not change the definition of **P** (see Arora, Barak [9], p. 30).

Thus, since neither the number of tapes nor the way the inputs and outputs are presented (binary coding or natural numbers) significantly affect, we can assert that

$$(\forall i)(\exists j)(\forall x)\left[\widehat{P}_i(x) = P_j\left(h_0(x)\right)\right] \,\&\, (\forall j)(\exists i)(\forall\sigma)\left[P_j(\sigma) = \widehat{P}_i\left(h_1(\sigma)\right)\right]$$

and

$$(\forall i)(\exists j)(\forall x)(\forall A)\left[\widehat{\boldsymbol{M}}_i(A)(x) = \boldsymbol{M}_j(h_0(A))(h_0(x)))\right] \,\&$$
$$(\forall j)(\exists i)(\forall\sigma)(\forall\Xi)\left[\boldsymbol{M}_j(\Xi)(\sigma) = \widehat{\boldsymbol{M}}_i(h_1(\Xi))(h_1(\sigma))\right].$$

In [12], the existence of a homomorphic mapping from $\widehat{\mathbf{P}}$ to **P** and, vice versa, the existence of a homomorphic mapping from $\widehat{\mathbf{P}}$ to **P** (with respect to the relations in question) were proved.

Now we will prove that **P** and $\widehat{\mathbf{P}}$ are isomorphic (with respect to the relations in question).

Define the relations in **P** and $\widehat{\mathbf{P}}$.

Definition 3. (*i*) Let $P_i, P_j \in \mathbf{P}$. P_i is to the *left* of $P_j\left(P_i <_L P_j\right)$ if $(\exists\, \gamma \in \Lambda^*)(\forall\, \tau \leq \gamma)$ $[P_i(\tau) = P_j(\tau) \,\&\, P_i(\gamma') < P_j(\gamma')]$ (i.e. $P_i(\gamma') = 0 \,\&\, P_j(\gamma') = 1$);

(*ii*) Let $\widehat{P}_i, \widehat{P}_j \in \widehat{\mathbf{P}}$. $\widehat{P}_i$ is to the *left* of $\widehat{P}_j\left(\widehat{P}_i <_L \widehat{P}_j\right)$ if $(\exists\, x)(\forall\, y < x)$ $\left[\widehat{P}_i(y) = \widehat{P}_j(y) \,\&.\, \widehat{P}_i(x+1) < \widehat{P}_j(x+1)\right]\left(\text{i.e.}, \widehat{P}_i(x+1) = 0 \,\&\, \widehat{P}_j(x+1) = 1\right)$.

It is shown in [12] that there is a homomorphic mapping from **P** to $\widehat{\mathbf{P}}$ and vice versa, there is a homomorphic mapping from $\widehat{\mathbf{P}}$ to **P** (with respect to the relations in question).

Proposition 1. The classes **P** and $\widehat{\mathbf{P}}$ are isomorphic.

Let's define the mapping $\varrho\colon \omega \to \omega$.

Let j_0 be such that $(\forall\sigma)\left[P_0(\sigma) = \widehat{P}_{j_0}\left(h_1(\sigma)\right)\right]$.

(As noted above, for P_0 there exists such $\widehat{P}_{j_0}$)

1) Define $\varrho(0) = j_0$.

n+1) Suppose that $(\forall k_0 \leq n)(\forall k_1)\left(\varrho(k_0 \neq k_1)\right)$.

Let m_0 be such that $(\forall\sigma)\left[P_n(\sigma) = \widehat{P}_{m_0}\left(h_1(\sigma)\right)\right]$.

If $m_0 \notin \{\varrho(0), \varrho(1), \cdots \varrho(n-1)\}$, then define $\varrho(n) = m_0$.

If $m_0 \in \{\varrho(0), \varrho(1), \cdots \varrho(n-1)$, then let m_1 be such that $(\forall\sigma)\left[P_n(\sigma) = \widehat{P}_{m_1}\left(h_1(\sigma)\right)\right] \,\&$ $m_1 \notin \{\varrho(0), \varrho(1), \cdots \varrho(n-1)\}$. Such m exists because according to the Padding Lemma (see

Soare [2], p.15, Rogers [1], p. 22), $(\forall v_0)(\exists v \geq v_0)$ [v is the index of c.e. set (i.e. the domain of the p.c. function φ_v) such that $(\forall x)(W_v(x) = \hat{P}_m(x))$ and for all x $W_v(x)$ is computed in the same time as $\hat{P}_m(x)$].

Then define $\varrho(n) = m_1$. (Thus the definition of mapping ϱ is completed.)

Let P_i,P_j are such that P_i<P_j .

Then $(\exists\, \gamma \in \Lambda^*)(\forall\, \tau \leq \gamma)[P_i(\tau) = P_j(\tau)\ \&\ P_i(\tau) = \hat{P}_{\varrho(i)}\big(h_1(\tau)\big)\ \&\ P_i(\gamma') = 0\ \&\ P_j(\gamma') = 1\,]$. Since $P_i(\gamma') = \hat{P}_{\varrho(i)}\big(h_1(\gamma')\big) = 0$ and $P_j(\gamma') = \hat{P}_{\varrho(j)}\big(h_1(\gamma')\big) = 1$ then $\hat{P}_{\varrho(i)}\big(h_1(\gamma')\big) < \hat{P}_{\varrho(j)}\big(h_1(\gamma')\big)$.

As $(\forall\, \tau \leq \gamma)\big[\hat{P}_{\varrho(i)}\big(h_1(\tau)\big) = \hat{P}_{\varrho(j)}\big(h_1(\tau)\big)\big]$ then $\hat{P}_{\varrho(i)} < \hat{P}_{\varrho(j)}$ (according to Definition 3).

So, if $P_i < P_j$, then $\hat{P}_{\varrho(i)} < \hat{P}_{\varrho(j)}$.

Thus, there is a mapping $\mathbf{P}\to\hat{\mathbf{P}}$ such that it preserves the order, i.e., there is an isomorphic mapping from **P** to $\hat{\mathbf{P}}$.

Similarly, one can prove the existence of an isomorphic mapping from $\hat{\mathbf{P}}$ to **P**. So, we can say that the classes **P** and $\hat{\mathbf{P}}$ are isomorphic (with respect to the relations in question).

2.1. Premliminaries about P-T-mitoticity

Definition 4. Define $\Theta \leq_T^P \Xi$, if there exists an i such that $B = \boldsymbol{M}_i(A)$ (see Ladner [5], Ambos-Spies [6]).

Definition 5. Define $B \leq_T^{\hat{P}} A$ if there is an i such that that $B = \widehat{\boldsymbol{M}}_i(A)$.

Definition 6. A *splitting of* A is a pair A_1, A_2 of c.e. sets such that $A_1 \cap A_2 = \emptyset$. We sometimes will write $A = A_1 \sqcup A_2$ if A_1, A_2 is a splitting of A (see Downey, Stob [13], p. 4).

Definition 7. A c.e. set$A$ is *T-mitotic* if there is a splittingA_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$ (see Downey, Stob [13], p. 83, Lachlan [14], pp. 9-10).

Let us recall some information about T-autoreducibility.

Definition 8. We say that a partial recursive functional $\boldsymbol{\Psi}$ is an *autoreduction* if, for all X and n, the computation of $\boldsymbol{\Psi}(X, n)$ includes no question of the form "$n \in X$?". A set A is *T-autoreducible* if there exists an autoreduction $\boldsymbol{\Psi}$ such that $A = \boldsymbol{\Psi}(A)$ (see Trakhtenbrot [15], Ladner [16], p. 199).

From the definition of T-autoreducibility it follows:

A is T-autoreducible $\Leftrightarrow (\exists e)(\forall x)(\boldsymbol{\Phi}_e(A \cup \{x\})(x)) = A(x)) \Leftrightarrow$

$(\exists e)(\forall x)(\boldsymbol{\Phi}_e(A - \{x\})(x)) = A(x))$.

Ambos-Spies introduced the following notions:

a) A computable set Ξ is *P-T-mitotic* if there is a set $\Theta \in \mathbf{P}$ such that $\Xi \equiv_T^P \Xi \cap \Theta \equiv_T^P \Xi \cap \overline{\Theta}$. Otherwise, Ξ is *non-P-T-mitotic* (see Ambos-Spies [6], p. 4).

b) A computable set Ξ is *P-T-autoreducible* if for some $n \in \omega$ and every $\sigma \in \Lambda^*$, $\Xi\,(\sigma) = \boldsymbol{M}_n(\Xi - \{\sigma\})(\sigma)$ (see Ambos-Spies [6], p.19).

(Ambos-Spice prefers the expression "$\Xi\,(\sigma) = \boldsymbol{M}_n(\Xi - \{\sigma\})(\sigma)$" instead of the equivalent expression "$\Xi\,(\sigma) = \boldsymbol{M}_n(\Xi \cup \{\sigma\})(\sigma)$". For the sets of nonnegative numbers, the expression "$A(x) = \boldsymbol{M}_n(A \cup \{x\})(x)$" is used in the definition of T-autoreducibility, for example, in Downey, Slaman [17], p. 121.)

Ambos-Spies has proved that

(i) if Ξ is *P-T*-mitotic, then Ξ is *P-T*-autoreducible(see Ambos-Spies [6], p.19),

(ii) there is a computable set Ξ, which is *P-T*-autoreducible, but not *P-T*-mitotic (see Ambos-Spies [6], p. 21).

We represent the definitions of $\hat{P}$-*T*-mitoticity and $\hat{P}$-*T*-autoreducibility according to Ambos-Spies with slight modifications (see Ambos-Spies [6]).

Definition 9. A computable set A is $\hat{P}$-*T-autoreducible* if for some $n \in \omega$ and every $x \in \omega, A(x) = \hat{\boldsymbol{M}}_n(A \cup \{x\})(x)$.

Definition 10. A computable set A is $\hat{P}$-*T-mitotic* if there is a set $B \in \hat{\mathbf{P}}$ such that $A \equiv_T^{\hat{P}} A \cap B \equiv_T^{\hat{P}} A \cap \bar{B}$. Otherwise, A is *non-*$\hat{P}$*-T-mitotic*.

Let us give the definitions of index sets $T(\hat{P})M,\ AT(\hat{P})$.

Definition 11. $T(\hat{P})M$=$\{z \mid W_z$ is $\hat{P}$-T-mitotic$\}$,

$AT(\hat{P}) = \{$z $|W_z$ is $\hat{P}$-T- autoreducible$\}$=$\{z \mid (\exists i)(\forall x)[M_i(W_z \cup \{x\})(x) = W_z(x)]$ & $(W_z$ is computable$)\}$.

2. 2. Premliminaries about P-m- mitoticity

Definition 12. (Computing a function and running time)

Let $f: \Lambda^* \to \Lambda^*$ and let $T: \omega \to \omega$ be some functions, and let M be a Turing machine (TM). We say that *M computes* f *in* $T(n)$*-time* (we write $T(n)$-time instead of T-time, for emphasis that T is applied to the input length), if for every $\sigma \in \Lambda^*$, if M is initialized to the start configuration on input σ, then after at most $T(|\sigma|)$ steps it halts with $f(\sigma)$ written on its output tape.

We say that *M computes* f if it computes f in $T(n)$ time for some function $f: \omega \to \omega$. (see Arora, Barak [9], p. 17)

Definition 13. $\{f_n : n \epsilon \omega\}$ is the effective enumeration of ***PF*** (the class of deterministically polynomial time computable functions from Λ^* to Λ^*).

Ξ is *polynomial time many-one* (*P-m*) *reducible* to Θ ($\Xi \leq_m^P \Theta$), if for some n, $(\forall \sigma \in \Lambda^*)$ $\left(\Xi(\sigma) = \Theta\left(f_n(\sigma)\right)\right)$ (see Ambos-Spies [6], p.2).

By analogy, for arbitrary n we will define the function $\hat{f}_n: \omega \to \omega$.

Let $\{\varphi_i\}_{i \epsilon \omega}$ be the enumeration of the partial computable (p.c.) functions of one variable and T_j be the Turing machine which computes the p.c. function φ_j (see Soare [2], p.12, Rogers [1], p. 12). Remind, that $\hat{h}$ is a computable function from ω onto ω^2. Then we define (proceeding from the fact that $\hat{h}(i) = (i_0, i_1)$) the function $\hat{f}_i$ (for all i) as follows:

Definition 14. a) For arbitrary i let T_{i_0} be initialized to the start configuration on input x. Then define the function $\hat{f}_i$ as follows:

$$\hat{f}_i(x) = \begin{cases} \text{the total number of 1's, appearing anywhere on the} \\ \text{tape, after u-th step,} \quad \text{if } (\exists u \le q_{i_1}) \left(T_{i_0} \text{ stops at } u\text{-th step} \right); \\ \text{the total number of 1's, appearing anywhere on the} \\ \text{tape, just after } q_{i_1}\text{-th step,} \quad \text{otherwise .} \end{cases}$$

b) A is $\hat{P}$*-m-reducible* to B $\left(A \le_m^{\hat{P}} B\right)$, if $(\exists i)(\forall x)(\forall s_1 \ge q_{i_1}(x))(\exists s_2 \ge s_1)$.

Definition 15. A computable set Ξ is *P-m-mitotic* if Ξ is finite or cofinite if there is a set $\Theta \epsilon \boldsymbol{P}$ such that $\Xi \equiv_m^P \Xi \cap \Theta \equiv_m^P \Xi \cap \overline{\Theta}$ (see Ambos-Spies [6], p. 4).

Definition 16. A computable set Ξ is *P-m-autoreducible* if Ξ is finite, or cofinite, or if for some $f \epsilon \boldsymbol{PF}$, $\Xi \le_m \Xi$ via f and $(\forall \sigma \in \Lambda^*)$ $(f(\sigma) \ne \sigma)$ (see Ambos-Spies [6], p.19).

Definition 17. A computable set A is $\hat{P}$*-m-mitotic* if A is finite or cofinite if there is a set $B \epsilon \widehat{\boldsymbol{P}}$ such that $A \equiv_m^{\hat{P}} A \cap B \equiv_m^{\hat{P}} A \cap \bar{B}$ (see Ambos-Spies [6], p. 4).

Definition 18. A computable set A is $\hat{P}$*-m-autoreducible* if A is finite, or cofinite, or if $(\exists i)[A \le_m A$ via $\hat{f}_i$, and $(\forall x)$ $(\hat{f}_i(x) \ne x))$.

Definition 19. For any given class $\mathcal{E}$ of computably enumerable sets, let $IND_{\mathcal{E}} = \{z | W_z \in \mathcal{E}\}$. If $A = IND_{\mathcal{E}}$ for some $\mathcal{E}, A$ is called an *index set* (see Rogers [1], p. 324).

Let us give the definitions of index sets $M(\hat{P}\text{-}m)$, $A(\hat{P}\text{-}m)$.

Definition 20. $M\left(\hat{P}\text{-}m\right) = \{z \mid W_z$ is $\hat{P}\text{-}m$-mitotic$\}$,

$A(\hat{P}\text{-}m) = \{$z $|W_z$ is $\hat{P}\text{-}m$- autoreducible$\}$.

3. Results

To formulate the main results, we remind the following definitions:

Definition 21. A set A is Σ_n-complete (Π_n-complete) if $A \in \Sigma_n(\Pi_n)$ and $B \le_1 A$ for every $B \in \Sigma_n(\Pi_n)$ (it makes no difference whether we use "$B \le_m A$" or "$B \le_1 A$" in the definition of Σ_n-complete and Π_n-complete) (see Soare [2], p. 64).

Definition 22. $Rec = \{z | W_z$ is computable (recursive)$\}$, $Fin = \{z | W_z$ is finite$\}$, $Cof = \{z | \overline{W}_z$ is finite$\}$ (see Soare [2], p. 21).

It is known that Fin is Σ_2-complete, Cof and Rec are Σ_3-complete (see Soare [2], pp. 65-67, Rogers [1], pp. 327-328).

One of the approaches to the problem of lower bounds (called a *reducibility approach* in [1]) is to take certain distinguished sets as standard "*reference points*" and to obtain bounds on the level (and degree) of any other given set by establishing reducibility relationships between it and one or more of the reference sets. In most cases, we shall use sets complete in Σ_n or Π_n $(n > 0)$ as *reference sets*, and we shall use *m*-reducibility. The reducibility approach is particularly useful for getting lower bounds on level (and degree). In conjunction with the Tarski-Kuratowski algorithm (and the strong hierarchy theorem), it sometimes enables us to identify not only the level but, indeed, the recursive-isomorphism type of a given set (see Rogers [1], p. 325).

Lemma 1. *Let* $\mathcal{E}$ *be the class of computably enumerable sets, such that* $IND_{\mathcal{E}} \supseteq Cof$, $\overline{IND_{\mathcal{E}}} \supseteq \overline{Rec}$ *and* $IND_{\mathcal{E}} \in \Sigma_3$. *Then* $IND_{\mathcal{E}}$ *is* Σ_3*-complete* (*note that* $\overline{Rec} = \{z \mid W_z$ *is non-computable*$\}$).

Proof. To prove Lemma 1, we use Rogers' proof of index set *Rec's* Σ_3-completeness (see Rogers [1], pp. 327-328). To do this, the Σ_3-complete *reference set B* is used (where $B = \{x \mid (\exists y)[y \in W_x \;\&\; W_x$ is infinite$]\}$) and it is proved that $B \leq_m Cof$ (namely, such a general computable function g is constructed that $[z \in B \iff$ g(z)$\in Rec]$. Moreover, the construction is such that eventually $[z \in B \implies$ g(z)$\in Cof]$ and $[z \notin B \implies$ g(z)$\in \overline{Rec}]$.

Thus, if the class $\mathcal{E}$ satisfies the requirements of Lemma 1, the abovementioned function g will m-reduce B to $IND_{\mathcal{E}}$ (i.e., $z \in B \iff$ g(z) $\in IND_{\mathcal{E}}$). And since $IND_{\mathcal{E}} \in \Sigma_3$, then $IND_{\mathcal{E}}$ is Σ_3-complete. □

In the article [12] it is proved that $AT(\hat{P})$ and $T(\hat{P})M$ are Σ_3-complete.

Theorem 1. $A(\hat{P}\text{-}m)$ *is* Σ_3*-complete.*

Proof. Let's first prove that $A(\hat{P}\text{-}m) \in \Sigma_3$.

$z \in A(\hat{P}\text{-}m) \iff [W_z$ is computable$]\;\&\; [[(\exists i)[W_z \leq_m^{\hat{P}} W_z$ via $\hat{f}_i \;\&\; (x) \neq x] \vee (W_z$is finite$) \vee$
$(W_z$ is cofinite$)] \iff (\exists z_1)(\forall n)(\forall u_0)\,(\exists u_1 \geq u_0)[(n \in W_{z,u_1} \;\&\; n \notin W_{z_1,u_1}) \vee$
$(n \notin W_{z,u_1} \;\&\; n \in W_{z_1,u_1}) \;\&\; [[(\exists i)(\forall x)(\forall s_1 \geq q_{i_1}(x))(\exists s_2 \geq s_1)$
$[W_{z,s_2}(x) = W_{z,s_2}\left(\hat{f}_{i,s_2}(x)\right) \;\&\; \hat{f}_{i,s_2}(x) \neq x] \vee\; (\exists t_0)(\forall t_1)[t_1 \leq t_0 \;\vee$
$W_{z,t_1} = W_{z,t_0} \;\vee\; (\exists v_0))(\forall v_1)(\exists t_2)[v_1 \leq v_0) \;\vee\; v_1 \in W_{z,t_2}]] \iff$
$(\exists z_1)(\exists i)(\exists t_0)(\exists v_0)$
$(\forall n)(\forall u_0)(\forall x)(\forall s_1 \geq q_{i_1}(x))(\forall t_1)(\forall v_1)$
$(\exists u_1 \geq u_0)(\exists s_2 \geq s_1)(\exists t_2)$
$[[(n \in W_{z,u_1} \;\&\; n \notin W_{z_1,u_1}) \vee (n \notin W_{z,u_1} \;\&\; n \in W_{z_1,u_1}) \;\&$
$[[\,W_{z,s_2}(x) = W_{z,s_2}\left(\hat{f}_{i,s_2}(x)\right) \;\&\; \hat{f}_{i,s_2}(x) \neq x] \vee$
$[t_1 \leq t_0 \;\vee\; W_{z,t_1} = W_{z,t_0}] \vee [v_1 \leq v_0 \;\vee\; v_1 \in W_{z,t_2}]]]$.
Thus, $T(\hat{P})M \in \Sigma_3$.

Since $A(\hat{P}\text{-}m) \supseteq Cof$, $\overline{A(\hat{P}\text{-}m)} \supseteq \overline{Rec}$ and $A(\hat{P}\text{-}m) \in \Sigma_3$ it follows from Lemma 1 that $A(\hat{P}\text{-}m)$ is Σ_3-complete. □

Theorem 2. $M(\hat{P}\text{-}m)$ *is* Σ_3*-complete.*

Proof. Let's first prove that $M(\hat{P}\text{-}m) \in \Sigma_3$.

$z \in M(\hat{P}\text{-}m) \iff [W_z$ is computable$]\;\&\; [[(\exists i_0)[W_z \equiv_m^{\hat{P}} (W_z \cap \hat{P}_{i_0}) \equiv_m^{\hat{P}} (W_z \cap \bar{\hat{P}}_{i_0})\,]$
$\vee (W_z$ is finite$) \;\vee (W_z$ is cofinite$)] \iff$
$(\exists z_1)(\forall n)(\forall u_0)\,(\exists u_1 \geq u_0)[(n \in W_{z,u_1} \;\&\; n \notin W_{z_1,u_1}) \vee\; (n \notin W_{z,u_1} \;\&\; n \in W_{z_1,u_1}) \;\&$
$[[(\exists i)(\exists j)(\forall x_1)(\forall s_1 \geq max(q_{i_1}(x_1), q_{j_1}(x_1)))(\exists s_2 \geq s_1)$
$[W_{z,s_2}(x_1) = (W_{z,s_2} \cap \hat{P}_{i,s_2})\left(\hat{f}_{j,s_2}(x_1)\right)] \;\&$
$(\exists k)(\forall x_2)(\forall s_3 \geq max(q_{i_1}(x_2), q_{k_1}(x_2)))(\exists s_4 \geq s_3)$

$[(W_{z,s_4} \cap \hat{P}_{i,s_4})(x_2) = W_{z,s_4}\left(\hat{f}_{k,s_4}(x_2)\right)]$ &
$(\exists l)(\forall x_3)(\forall s_5 \geq max(q_{i_1}(x_3), q_{l_1}(x_3)))(\exists s_6 \geq s_5)$
$[W_{z,s_6}(x_3) = (W_{z,s_6} \cap \bar{\hat{P}}_{i,s_6})\left(\hat{f}_{l,s_6}(x_3)\right)]$ &
$(\exists m)(\forall x_4)(\forall s_7 \geq max(q_{i_1}(x_4), q_{m_1}(x_4)))(\exists s_8 \geq s_7)$
$[(W_{z,s_8} \cap \bar{\hat{P}}_{i,s_8})(x_4) = W_{z,s_8}\left(\hat{f}_{m,s_8}(x_4)\right)]$ $\vee$
$(\exists t_0)(\forall t_1)[t_1 \leq t_0 \vee W_{z,t_1} = W_{z,t_0}] \vee (\exists v_0)(\forall v_1)(\exists t_2)[v_1 \leq v_0 \vee v_1 \in W_{z,t_2}]] \Leftrightarrow$
$(\exists z_1)(\exists i)(\exists j)(\exists k)(\exists l)(\exists m)(\exists t_0)(\exists v_0)$
$(\forall n)(\forall u_0)(\forall x_1)(\forall s_1 \geq max(q_{i_1}(x_1), q_{j_1}(x_1)))(\forall x_2)(\forall s_3 \geq max(q_{i_1}(x_2), q_{k_1}(x_2)))$
$(\forall x_3)(\forall s_5 \geq max(q_{i_1}(x_3), q_{l_1}(x_3)))(\forall x_4)(\forall s_7 \geq max(q_{i_1}(x_4), q_{m_1}(x_4)))(\forall t_1)(\forall v_1)$
$(\exists u_1 \geq u_0)(\exists s_2 \geq s_1)(\exists s_4 \geq s_3)(\exists s_6 \geq s_5)(\exists s_8 \geq s_7)(\exists t_2)$
$[(n \in W_{z,u_1} \& \ n \notin W_{z_1,u_1}) \vee (n \notin W_{z,u_1} \& \ n \in W_{z_1,u_1})$ &
$[W_{z,s_2}(x_1) = (W_{z,s_2} \cap \hat{P}_{i,s_2})(\hat{f}_{j,s_2}(x_1))]$ &.
$[(W_{z,s_4} \cap \hat{P}_{i,s_4})(x_2) = W_{z,s_4}\left(\hat{f}_{k,s_4}(x_2)\right)]$ &
$[W_{z,s_6}(x_3) = (W_{z,s_6} \cap \bar{\hat{P}}_{i,s_6})\left(\hat{f}_{l,s_6}(x_3)\right)]$ &
$[(W_{z,s_8} \cap \bar{\hat{P}}_{i,s_8})(x_4) = W_{z,s_8}\left(\hat{f}_{m,s_8}(x_4)\right)] \vee$
$[t_1 \leq t_0 \vee W_{z,t_1} = W_{z,t_0}] \vee [v_1 \leq v_0 \vee v_1 \in W_{z,t_2}]]$.

Thus, $M(\hat{P}\text{-}m) \in \Sigma_3$.

Since $M(\hat{P}\text{-}m) \supseteq Cof$, $\overline{M(\hat{P}\text{-}m)} \supseteq \overline{Rec}$ and $M(\hat{P}\text{-}m) \in \Sigma_3$, it follows from Lemma 1 that $M(\hat{P}\text{-}m)$ is Σ_3-complete.□

4. Conclusion

It is known that an effective enumeration of the sets of the class **P** (namely, $P_0, P_1, \cdots, P_i, \cdots$) exists and, thus, $\mathbf{P} = \{P_i | i \in \omega\}$. Based on the available numbering of computably enumerable sets $\{W_i\}_{i\in\omega}$, a sequence of sets of non-negative numbers $\hat{P}_i$ is constructed such that their effective enumeration exists and $\hat{\mathbf{P}} = \{\hat{P}_i \mid i \in \omega\}$ by definition.

It is shown that the class $\hat{\mathbf{P}}$ is isomorphic to the class **P**. Using traditional methods, it is shown that the index sets $A(\hat{P}\text{-}m)$ and $M(\hat{P}\text{-}m)$ are Σ_3-sets. Applying the method used by H.Rogers in proving the Σ_3-completeness of the index set $\{z \mid W_z$ is computable$\}$, it is proved that the index sets $A(\hat{P}\text{-}m) = \{z \mid W_z$ is $\hat{P}$-m-autoreducible$\}$ and $M(\hat{P}\text{-}m) = \{z | W_z$ is $\hat{P}$-m-mitotic$\}$ are Σ_3-complete.

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P-*m*-միթոտիկ բազմություններ և թվաբանական աստիճանակարգ

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Ամփոփում

Դիցուք $\{0,1\}^*$-ը $\{0,1\}$ բազմության տարրերից կազմված բոլոր վերջավոր շղթաների բազմություն է և **P**-ն այնպիսի *հիմնախնդիրների* դաս է, որոնք ճանաչելի են դետերմինիստական Թյուրինգյան մեքենաների միջոցով, որոնց աշխատանքի ժամանակը բազմանդամորեն է կախված մուտքային տվյալների չափից (*հիմնախնդիրը* փաստորեն $\{0,1\}^*$ բազմության ենթաբազմություն է):

Սույն հոդվածում սահմանված է $\widehat{\mathbf{P}}$ դասը և ցույց է տրված, որ $\widehat{\mathbf{P}}$-ն իզոմորֆ է **P** դասին:

Ելնելով *T*-միթոտիկություն և *T*-ինքնահանգեցում հասկացություններից Կ.Ամբոս-Սպիսը ներմուծել է *P*-*m*-միթոտիկություն և *P*-*m*-ինքնահանգեցում հասկացությունները:

Համանմանորեն ներմուծվել են $\widehat{\mathrm{P}}$-*m*-միթոտիկություն և $\hat{P}$-*m*-ինքնահանգեցում հասկացությունները:

Տվյալ հոդվածում ապացուցված է, որ $\{z \mid W_z$-ն $\widehat{\mathrm{P}}$-*m*-միթոտիկ է$\}$ և $\{z \mid W_z$-ն $\widehat{\mathrm{P}}$-*m*-ինքնահանգեցվող է$\}$ ինդեքսային բազմությունները Σ_3-լրիվ են.

Բանալի բառեր՝ Թվաբանական աստիճանակարգ, *P*-*m*-միթոտիկ բազմություն, *P*-*m*- ինքնահանգեցվող բազմություն, ինդեքսային բազմություն:

P-m-митотические множества и арифметическая иерархия

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Аннотация

Пустъ $\{0,1\}^*$ является множеством всех конечных цепочек, составленных из элементов множества $\{0,1\}$ и $\mathbf{P}$ является классом *проблем*, распознаваемых детерминированными машинами Тьюринга, время работы которых полиномиально зависит от размера входных данных (*проблема* фактически является подмножеством множества $\{0,1\}^*$).

В данной статье определен класс $\widehat{\mathbf{P}}$ и показано, что $\widehat{\mathbf{P}}$ изоморфен классу $\mathbf{P}$.

Исходя из понятий T- митотичности и T-автосводимости К. Амбос-Спис ввел понятия P-m- митотичности и P-m- автосводимости.

По аналогии с упомянутыми понятиями введены понятия $\widehat{\mathrm{P}}$-m-митотичности и $\widehat{\mathrm{P}}$-m-автосводимости.

В данной статье доказано, что индексные множества, $\{z \mid W_z - \widehat{\mathrm{P}}\text{-}m\text{-митотично}\}$ и $\{z \mid W_z - \widehat{\mathrm{P}}\text{-}m\text{-автосводимо}\}$ являются Σ_3-полными.

Ключевые слова` Арифметическая иерархия, P-m-митотическое множество, P-m-автосводимое множество, индексное множество.