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Analytical Inversion of Tridiagonal Hermitian Matrices

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Abstract

In this paper we give an algorithm for inverting complex tridiagonal Hermitian matrices with optimal computational efforts. For matrices of a special form and, in particular, for Toeplitz matrices, the derived formulas lead to closed-form expressions for the elements of inverse matrices.
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1. Introduction

Tridiagonal matrices are encountered in many areas of applied mathematics. Such matrices are of great importance in finite difference and finite element methods for differential equations. The construction of cubic splines is reduced to solving systems with tridiagonal matrices. Symmetric matrices are reduced to tridiagonal matrices by the similarity Householder transformation (see [1, 2, 3], for instance). Other examples can be cited.

There is a well-known fast numerical method for solving systems with tridiagonal matrices. At the same time, the analytical matrix inversion is also of certain interest (see [4, 5, 6], for instance). For tridiagonal matrices of special types, this leads to closed-form expressions for the elements of inverse matrices [7, 8, 9, 10]. This is undoubtedly useful in theoretical considerations. Further, explicit formulas can be a part of more general computational procedures. There are other reasons as well.

In this article, we focus our attention on complex Hermitian tridiagonal matrices. We will construct a fairly simple computational procedure, consisting of a sequence of recurrence relations, leading to the calculation of the elements of the inverse matrix. In special cases, in particular for Toeplitz tridiagonal Hermitian matrices, the procedure can become the basis for deriving closed-form expressions for the elements of the inverse matrix.

We note right away that throughout this article $\overline{z}$ stands for the complex conjugate of the complex number z.

Let a nonsingular tridiagonal Hermitian matrix

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ \overline{b_1} & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \overline{b_{n-2}} & a_{n-1} & b_{n-1} \\ & & & \overline{b_{n-1}} & a_n \end{bmatrix} \tag{1}$$

be given, where a_i, $i = 1, 2, \ldots, n$ are real numbers and $b_i \neq 0$ for $i = 1, 2, \ldots, n-1$. In accordance with the accepted notation, $A = A^*$. We assume that $n > 3$. The requirement that the subdiagonal (superdiagonal) elements of the matrix be nonzero is not restrictive. Indeed, if some of these elements are equal to zero, the problem of computing the inverse matrix is decomposed into several similar problems for tridiagonal matrices of lower order.

2. Preliminary Calculations

Let $A^{-1} = [x_{ij}]_{n\times n}$. This matrix is also Hermitian. In our considerations we will use the notation

$$X^{(j)} \equiv [x_{1j}\, x_{2j} \ldots x_{nj}]^T, \quad j = 1, 2, \ldots, n$$

for the columns of the inverse matrix.

The matrix A can be represented as a product

$$A = DB \tag{2}$$

of the matrices

$$D = diag\,[b_1, \overline{b_1}, \overline{b_2}, \ldots, \overline{b_{n-2}}, \overline{b_{n-1}}] \tag{3}$$

and

$$B = \begin{bmatrix} p & 1 & & & & \\ 1 & f_2 & g_2 & & 0 & \\ & 1 & f_3 & g_3 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & f_{n-1} & g_{n-1} \\ & & & & 1 & q \end{bmatrix}, \tag{4}$$

where

$$f_i = \frac{a_i}{\overline{b_{i-1}}},\; g_i = \frac{b_i}{\overline{b_{i-1}}},\; i = 2, 3, \ldots, n-1;\; p = \frac{a_1}{b_1},\; q = \frac{a_n}{\overline{b_{n-1}}}. \tag{5}$$

Having a nonsingular matrix B defined in (4), let us consider the following system of linear algebraic equations

$$\begin{aligned} &p\mu_1 + \mu_2 = \alpha \\ &\mu_{i-1} + f_i\mu_i + g_i\mu_{i+1} = 0,\;\; 2 \le i \le n-1 \\ &\mu_{n-1} + q\mu_n = 0, \end{aligned} \tag{6}$$

where we will set the right-hand side α of the first equation a little later. It is easy to verify that regardless of the choice of α, the recursively defined quantities

$$\begin{aligned} &\mu_n = 1\,,\; \mu_{n-1} = -q\,, \\ &\mu_{i-1} = -f_i\mu_i - g_i\mu_{i+1}\,,\; i = n-1, n-2, \ldots, 2 \end{aligned} \tag{7}$$

satisfy all equations of the system (6), starting with the second one. Then, we choose the quantity α as follows:

$$\alpha = p\mu_1 + \mu_2. \tag{8}$$

Remark 1 *Since, by assumption, the matrix B is nonsingular (it follows from (2)), then $\alpha \neq 0$. Indeed, otherwise we would have obtained that the homogeneous system (6) has a nontrivial solution. Further,*

$$\alpha = \frac{a_1}{b_1}\mu_1 + \mu_2 = \frac{1}{b_1}(a_1\mu_1 + b_1\mu_2).$$

Therefore

$$a_1\mu_1 + b_1\mu_2 \neq 0$$

as well.

Thus,

$$\alpha = b_1^{-1} t^{-1}, \tag{9}$$

where

$$t \equiv (a_1\mu_1 + b_1\mu_2)^{-1}. \tag{10}$$

Let us introduce the vector

$$r^{(1)} \equiv [\mu_1\, \mu_2 \dots \mu_n]^T,$$

the components of which are specified in (7). As follows from (4), (6) and (9),

$$Br^{(1)} = [\alpha\, 0 \dots 0]^T = \alpha e^{(1)} = b_1^{-1} t^{-1} e^{(1)},$$

where $e^{(1)} \equiv [1\, 0 \dots 0]^T$. Further, on the basis of factorization (2) of the matrix A, we obtain the equality

$$Ar^{(1)} = DBr^{(1)} = b_1^{-1} t^{-1} De^{(1)} = t^{-1} e^{(1)}; \tag{11}$$

here we have used the obvious equality $De^{(1)} = b_1 e^{(1)}$ (see (3)). The equality (11) allows to compute the first column of the inverse matrix A^{-1}. Indeed, from this equality we find that

$$A^{-1} e^{(1)} = tr^{(1)}.$$

Since $A^{-1}e^{(1)} = X^{(1)}$, then $X^{(1)} = tr^{(1)}$, or

$$x_{i1} = t\mu_i, \quad i = 1, 2, \dots, n. \tag{12}$$

Thus, we have found the first column of the inverse matrix. Similarly, we can calculate the last column of the matrix A^{-1}. For this purpose, let us consider the linear system

$$\begin{aligned} &p\nu_1 + \nu_2 = 0 \\ &\nu_{i-1} + f_i\nu_i + g_i\nu_{i+1} = 0, \;\; 2 \leq i \leq n-1 \\ &\nu_{n-1} + q\nu_n = \beta, \end{aligned} \tag{13}$$

where we will set the right-hand side β of the last equation later. Regardless of the choice of β, the recursively defined quantities

$$\begin{aligned} &\nu_1 = 1\,, \;\; \nu_2 = -p\,, \\ &\nu_{i+1} = -\frac{1}{g_i}(\nu_{i-1} + f_i\nu_i)\,, \;\; i = 2, 3, \dots, n-1 \end{aligned} \tag{14}$$

satisfy the first $n-1$ equations of the system (13). Then we choose the quantity β as follows:

$$\beta = \nu_{n-1} + q\nu_n. \tag{15}$$

Since the matrix B is nonsingular, then $\beta \neq 0$ (see Remark 1). Substituting the expression of the quantity q given in (5) into (15) yields

$$\beta = \nu_{n-1} + \frac{a_n}{\overline{b_{n-1}}}\nu_n = \frac{1}{\overline{b_{n-1}}}(\overline{b_{n-1}}\nu_{n-1} + a_n\nu_n).$$

Thus,

$$\beta = \overline{b_{n-1}}^{-1}\theta^{-1}, \tag{16}$$

where

$$\theta \equiv (\overline{b_{n-1}}\nu_{n-1} + a_n\nu_n)^{-1}.$$

Now let us introduce the vector

$$r^{(n)} \equiv [\nu_1\, \nu_2 \dots \nu_n]^T,$$

the components of which are specified in (14). From (4), (13) and (16) we find that

$$Br^{(n)} = [0, \dots 0\, \beta]^T = \beta e^{(n)} = \overline{b_{n-1}}^{-1}\theta^{-1}e^{(n)},$$

where $e^{(n)} \equiv [0 \dots 0\, 1]^T$. Having the factorization (2) of the matrix A, we obtain the equality

$$Ar^{(n)} = DBr^{(n)} = \overline{b_{n-1}}^{-1}\theta^{-1}De^{(n)} = \theta^{-1}e^{(n)}.$$

From here,

$$A^{-1}e^{(n)} = \theta r^{(n)}.$$

Since $A^{-1}e^{(n)} = X^{(n)}$, then $X^{(n)} = \theta r^{(n)}$, or

$$x_{in} = \theta\nu_i, \quad i = 1, 2, \dots, n. \tag{17}$$

Let us refine the last expression. From (12), $x_{n1} = t\mu_n = t$. Further, according to (17), $x_{1n} = \theta\nu_1 = \theta$. Since A^{-1} is a Hermitian matrix, then $x_{1n} = \overline{x_{n1}}$. Consequently, $\theta = \bar{t}$, and we come to the conclusion that

$$x_{in} = \bar{t}\nu_i, \quad i = 1, 2, \dots, n. \tag{18}$$

So, we have found the first and the last columns of the Hermitian matrix A^{-1}. These are expressions (12) and (18). Taking into account that $\nu_1 = 1$ and $\mu_n = 1$, we write these elements in the form of

$$x_{i1} = t\mu_i\overline{\nu_1}, \ x_{in} = \bar{t}\,\overline{\mu_n}\nu_i, \quad i = 1, 2, \dots, n. \tag{19}$$

Moreover, the diagonal elements $x_{11} = t\mu_1\overline{\nu_1}$ and $x_{nn} = \bar{t}\,\overline{\mu_n}\nu_n$ are real numbers. Therefore, we can write $x_{nn} = t\mu_n\overline{\nu_n}$ as well.

Looking ahead, we say that in the next section we will prove that the quantities

$$t\mu_i\overline{\nu_i}, \quad i = 2, 3, \dots, n-1 \tag{20}$$

are the remaining diagonal elements of the matrix A^{-1}. To do this, here we first establish that the quantities (20) are real numbers (naturally, without assuming that they are somehow related to the matrix A^{-1}).

Let us introduce into consideration the quantities

$$R_i \equiv b_{i-1}(t\mu_i\overline{\nu_{i-1}}) + \overline{b_{i-1}}(t\mu_{i-1}\overline{\nu_i}), \quad i = 2, 3, \dots, n-2. \tag{21}$$

Lemma 1. *The quantity R_2 is a real number.*

Proof. Since $\nu_1 = 1$ and $\nu_2 = -p$ (see (2.13)), then

$$R_2 = t(b_1\mu_2\overline{\nu_1} + \overline{b_1}\mu_1\overline{\nu_2}) = tb_1(\mu_2 - p\mu_1).$$

Further, taking into account the equalities (8) and (9), we get

$$R_2 = tb_1(\alpha - 2p\mu_1) = tb_1\alpha - 2pb_1(t\mu_1) = 1 - 2a_1(t\mu_1).$$

The quantities a_1 and $t\mu_1$ are real numbers, so R_2 is also a real number. $\square$

Lemma 2. *The quantities R_i from (21) satisfy the relations*

$$R_i = -R_{i-1} - 2a_{i-1}(t\mu_{i-1}\overline{\nu_{i-1}}), \quad i = 3, 4, \ldots, n-2. \tag{22}$$

Proof. From (6) we have the equality

$$\mu_{i-2} + f_{i-1}\mu_{i-1} + g_{i-1}\mu_i = 0.$$

Using formulas (5), let us write this equality in the form of

$$\overline{b_{i-2}}\mu_{i-2} + a_{i-1}\mu_{i-1} + b_{i-1}\mu_i = 0.$$

Multiplying both parts of the last equality by $t\overline{\nu_{i-1}}$, we get that

$$b_{i-1}(t\mu_i\overline{\nu_{i-1}}) = -\overline{b_{i-2}}(t\mu_{i-2}\overline{\nu_{i-1}}) - a_{i-1}(t\mu_{i-1}\overline{\nu_{i-1}}). \tag{23}$$

Similarly, from (13) we have the equality

$$\nu_{i-2} + f_{i-1}\nu_{i-1} + g_{i-1}\nu_i = 0,$$

which can be written as follows:

$$b_{i-2}\overline{\nu_{i-2}} + a_{i-1}\overline{\nu_{i-1}} + \overline{b_{i-1}}\overline{\nu_i} = 0.$$

Multiplying both parts of this equality by $t\mu_{i-1}$ yields

$$\overline{b_{i-1}}(t\mu_{i-1}\overline{\nu_i}) = -b_{i-2}(t\mu_{i-1}\overline{\nu_{i-2}}) - a_{i-1}(t\mu_{i-1}\overline{\nu_{i-1}}). \tag{24}$$

The relation (22) follows directly from the equalities (23) and (24). $\square$

Lemma 3. *The quantities $t\mu_i\overline{\nu_i}$, $\quad i = 2, 3, \ldots, n-1$ are real numbers.*

Proof. Consider first the quantity $t\mu_2\overline{\nu_2}$. Since $p\mu_1 + \mu_2 = \alpha$ and $\nu_2 = -p$ (see (6) and (14)), then

$$t\mu_2\overline{\nu_2} = t(p\mu_1 - \alpha)\overline{p} = (p\overline{p})(t\mu_1) - t\alpha\overline{p}.$$

Further, using the equality (9), we obtain that

$$t\mu_2\overline{\nu_2} = (p\overline{p})(t\mu_1) - \frac{\overline{p}}{b_1} = (p\overline{p})(t\mu_1) - \frac{a_1}{b_1\overline{b_1}}.$$

Thus, the quantity $t\mu_2\overline{\nu_2}$ is a real number.

Next, consider the quantity $t\mu_3\overline{\nu_3}$. As follows from (6) and (13),

$$\mu_3 = -\frac{a_2}{b_2}\mu_2 - \frac{\overline{b_1}}{b_2}\mu_1, \quad \overline{\nu_3} = -\frac{a_2}{\overline{b_2}}\overline{\nu_2} - \frac{b_1}{\overline{b_2}}\overline{\nu_1}.$$

Proceeding from these equalities, we get that

$$t\mu_3\overline{\nu_3} = \frac{1}{b_2\overline{b_2}}\left[a_2^2(t\mu_2\overline{\nu_2}) + b_1\overline{b_1}(t\mu_1\overline{\nu_1}) + a_2R_2\right].$$

The quantities $t\mu_1\overline{\nu_1}$ and $t\mu_2\overline{\nu_2}$ are real numbers. According to Lemma 1, the quantity R_2 is also a real number. Therefore, $t\mu_3\overline{\nu_3}$ is a real number as well.

Further reasoning will be carried out by the method of mathematical induction on i. Suppose that for some value of i, where $3 \le i \le n-2$, it is already known that the quantities $t\mu_k\overline{\nu_k}$, $k \le i$ and R_k, $k \le i-1$ are real numbers. From (6) and (13) we have

$$\mu_{i+1} = -\frac{a_i}{b_i}\mu_i - \frac{\overline{b_{i-1}}}{b_i}\mu_{i-1}, \quad \overline{\nu_{i+1}} = -\frac{a_i}{\overline{b_i}}\overline{\nu_i} - \frac{b_{i-1}}{\overline{b_i}}\overline{\nu_{i-1}}.$$

Then

$$t\mu_{i+1}\overline{\nu_{i+1}} = \frac{1}{b_i\overline{b_i}}\left[a_i^2(t\mu_i\overline{\nu_i}) + b_{i-1}\overline{b_{i-1}}(t\mu_{i-1}\overline{\nu_{i-1}}) + a_iR_i\right].$$

Hence, by virtue of the assumptions made and taking into account the assertion of Lemma 2, we arrive at a conclusion that the quantity $t\mu_{i+1}\overline{\nu_{i+1}}$ is a real number. □

Remark 2 *We have established that the quantities $t\mu_i\overline{\nu_i}$, $i = 1, 2, \ldots, n$ are real numbers. Therefore, $t\mu_i\overline{\nu_i} = \overline{t}\,\overline{\mu_i}\nu_i$.*

3. The Elements of the Inverse Matrix

Above we obtained the expressions (19) for the elements of the first and the last columns of the inverse matrix, as well as some auxiliary statements. Based on these results, here we derive formulas for the remaining elements of the inverse matrix.

Let $2 \le j \le n-1$. We introduce into consideration the vector

$$r^{(j)} \equiv [\overline{t}\,\overline{\mu_j}\nu_1, \ldots, \overline{t}\,\overline{\mu_j}\nu_{j-1}, t\mu_j\overline{\nu_j}, t\mu_{j+1}\overline{\nu_j}, \ldots, t\mu_n\overline{\nu_j}]^T, \tag{25}$$

where the quantities μ_i and ν_i are specified in (7) and (14), respectively. Multiplying the matrix B defined in (4) and the vector $r^{(j)}$ yields

$$Br^{(j)} = z^{(j)}, \tag{26}$$

where the components of the vector

$$z^{(j)} = [z_1^{(j)}\ z_2^{(j)} \ldots z_{j-1}^{(j)}\ \delta_j\ z_{j+1}^{(j)} \ldots z_{n-1}^{(j)}\ z_n^{(j)}]^T$$

are calculated as follows:

$$\begin{aligned}
z_1^{(j)} &= \overline{t}\,\overline{\mu_j}(p\nu_1 + \nu_2),\\
z_i^{(j)} &= \overline{t}\,\overline{\mu_j}(\nu_{i-1} + f_i\nu_i + g_i\nu_{i+1}), \quad 2 \le i \le j-1,\\
\delta_j &= \overline{t}\,\overline{\mu_j}\nu_{j-1} + f_j(t\mu_j\overline{\nu_j}) + g_j(t\mu_{j+1}\overline{\nu_j}),\\
z_i^{(j)} &= t(\mu_{i-1} + f_i\mu_i + g_i\mu_{i+1})\overline{\nu_j}, \quad j+1 \le i \le n-1,\\
z_n^{(j)} &= t(\mu_{n-1} + q\mu_n)\overline{\nu_j}.
\end{aligned}$$

Having equations (6) and (13), we conclude that $z_i^{(j)} = 0$ for $1 \le i \le j-1$ and $j+1 \le i \le n$. Thus,

$$z^{(j)} = [0 \,\dots\, 0\, \delta_j\, 0 \,\dots\, 0]^T = \delta_j e^{(j)}, \tag{27}$$

where $e^{(j)} = [0 \dots 0\, 1\, 0 \dots 0]^T$ (the unit is located on jth place).

It remains to clarify the quantity δ_j. Taking into account Remark 2, we have

$$\begin{aligned} \delta_j &= \bar{t}\,\overline{\mu_j}\nu_{j-1} + f_j(\bar{t}\,\overline{\mu_j}\nu_j) + g_j(t\mu_{j+1}\overline{\nu_j}) \\ &= \bar{t}\,\overline{\mu_j}(\nu_{j-1} + f_j\nu_j) + g_j(t\mu_{j+1}\overline{\nu_j}). \end{aligned} \tag{28}$$

Since $\nu_{j-1} + f_j\nu_j = -g_j\nu_{j+1}$ (see (13)), then

$$\delta_j = g_j(t\mu_{j+1}\overline{\nu_j} - \bar{t}\,\overline{\mu_j}\nu_{j+1}), \quad 2 \le j \le n-1. \tag{29}$$

Let us get one more representation of the quantity δ_j. Since $g_j\mu_{j+1} = -\mu_{j-1} - f_j\mu_j$ (see (6)), then from(28) it follows that

$$\delta_j = \bar{t}\,\overline{\mu_j}\nu_{j-1} - t\mu_{j-1}\overline{\nu_j} + f_j(\bar{t}\,\overline{\mu_j}\nu_j - t\mu_j\overline{\nu_j}).$$

From here, according to Remark 2, we obtain

$$\delta_j = \bar{t}\,\overline{\mu_j}\nu_{j-1} - t\mu_{j-1}\overline{\nu_j}, \quad 2 \le j \le n-1. \tag{30}$$

Assuming that $3 \le j \le n-1$, we can write the expression (30) in the form of

$$\delta_j = \frac{1}{\overline{g_{j-1}}}\,\overline{g_{j-1}(t\mu_j\overline{\nu_{j-1}} - \bar{t}\,\overline{\mu_{j-1}}\nu_j)}.$$

Comparing with the record (29), we arrive at the relation

$$\delta_j = \frac{1}{\overline{g_{j-1}}}\,\overline{\delta_{j-1}}, \quad 3 \le j \le n-1. \tag{31}$$

Based on the relation (31), one can easily show that

$$\delta_j = \begin{cases} \overline{b_{j-1}}^{-1} b_1 \overline{\delta_2}\,, & \text{if } j \text{ is odd,} \\ \overline{b_{j-1}}^{-1} \overline{b_1} \delta_2\,, & \text{if } j \text{ is even.} \end{cases} \tag{32}$$

Finally, let us calculate the quantity δ_2. According to the representation (30), we have

$$\begin{aligned} \delta_2 &= \bar{t}\,\overline{\mu_2}\nu_1 - t\mu_1\overline{\nu_2} = \bar{t}\,\overline{\mu_2} + t\mu_1\overline{p} \\ &= \bar{t}\,\overline{\mu_2} + \bar{t}\,\overline{\mu_1}\,\overline{p} = \bar{t}\,(\overline{\mu_2} + \overline{p}\,\overline{\mu_1}) = \bar{t}\,\overline{\alpha} = \overline{b_1}^{-1}, \end{aligned} \tag{33}$$

(see (6) and (9)). Thus, from (32) and (33) we conclude that

$$\delta_j = \overline{b_{j-1}}^{-1}, \quad j = 2, 3, \dots, n-1. \tag{34}$$

Summing up the results, from (27) and (34) we come to the equality

$$z^{(j)} = \overline{b_{j-1}}^{-1} e^{(j)}. \tag{35}$$

Proceeding from the factorization (2) of the matrix A and using the equalities (26) and (35), we have

$$Ar^{(j)} = DBr^{(j)} = Dz^{(j)} = \overline{b_{j-1}}^{-1} De^{(j)} = e^{(j)}$$

(note that $De^{(j)} = \overline{b_{j-1}} e^{(j)}$, which follows from (3)). Further,

$$A^{-1} e^{(j)} = r^{(j)}.$$

Since $A^{-1} e^{(j)} = X^{(j)}$, then $X^{(j)} = r^{(j)}$. The components of the vector $r^{(j)}$ are given in (25). Thus,

$$x_{ij} = \bar{t}\, \overline{\mu_j} \nu_i, \;\; i = 1, 2, \ldots, j-1 \quad \text{and} \quad x_{ij} = t \mu_i \overline{\nu_j}, \;\; i = j, j+1, \ldots, n. \tag{36}$$

Combining formulas (36) with those of (12) and (18) yields

$$x_{ij} = \begin{cases} \bar{t}\, \overline{\mu_j} \nu_i, \;\; i = 1, 2, \ldots, j-1, \\ t \mu_i \overline{\nu_j}, \;\; i = j, j+1, \ldots, n \end{cases} \qquad \text{for } j = 1, 2, \ldots, n. \tag{37}$$

Note the following. Since the matrix A^{-1} is also Hermitian, then in reality we only need to calculate the lower triangular part of this matrix.

Summarizing the considerations of Sections 2 and 3, let us write the following procedure to calculate the elements of the inverse matrix $A^{-1} = [x_{ij}]_{n \times n}$ for nonsingular matrix A given in (1).

Procedure Inv 3d Hermitian

1. Input elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$ of the matrix A (see (1)).
2. Calculate the quantities f_i, g_i, p and q (see (5)):

$$f_i = \frac{a_i}{\overline{\overline{b_{i-1}}}}, \;\; g_i = \frac{b_i}{\overline{\overline{b_{i-1}}}}, \;\; i = 2, 3, \ldots, n-1; \;\; p = \frac{a_1}{b_1}, \;\; q = \frac{a_n}{\overline{\overline{b_{n-1}}}}.$$

3. Calculate recursively the quantities μ_i (see (7)):

$$\mu_n = 1\,, \;\; \mu_{n-1} = -q\,,$$
$$\mu_i = -f_{i+1} \mu_{i+1} - g_{i+1} \mu_{i+2}\,, \;\; i = n-2, n-3, \ldots, 1.$$

4. Calculate recursively the quantities ν_i (see (14)):

$$\nu_1 = 1\,, \;\; \nu_2 = -p\,,$$
$$\nu_i = -\frac{1}{g_{i-1}} (\nu_{i-2} + f_{i-1} \nu_{i-1})\,, \;\; i = 3, 4, \ldots, n.$$

5. Calculate the quantity t (see (10) and Remark 1):

$$t = (a_1 \mu_1 + b_1 \mu_2)^{-1}.$$

6. Calculate the lower triangular part of the matrix A^{-1} (see (37)):

$$x_{ij} = t \mu_i \overline{\nu_j}, \; i = j, j+1, \ldots, n\,; \quad j = 1, 2, \ldots, n\,.$$

7. Set the upper triangular part of the matrix A^{-1} (see (37)):

$$x_{ij} = \overline{x_{ji}},\ i = 1, 2, \ldots, j-1\,; \quad j = 2, 3, \ldots, n\,.$$

8. Output the matrix $A^{-1} = [x_{ij}]_{n\times n}$.

End procedure

The procedure **Inv 3d Hermitian** can be useful for the following purposes. Firstly, it can be used as a basis of numerical algorithms for inverting nonsingular tridiagonal Hermitian matrices. In this case, it is easy to make sure that computing the lower triangular part of the matrix A^{-1} requires $0.5n^2 + O(n)$ arithmetical operations with complex numbers. Secondly, for matrices of special types, the procedure can be used for deriving closed form expressions for the elements of inverse matrices. The next section is devoted to this issue.

4. Toeplitz Tridiagonal Hermitian Matrices

Let us consider a matrix

$$A = \begin{bmatrix} a & b & & & \\ \bar{b} & a & b & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \bar{b} & a & b \\ & & & \bar{b} & a \end{bmatrix} \tag{38}$$

of order n, where a is a real number and $b \neq 0$. Additionally, we assume that

$$|a| \geq 2|b|. \tag{39}$$

Condition (39) ensures the nonsingularity of the matrix (38) (see [11], for instance).

For the matrix we are considering, the quantities calculated in Item 2 of the procedure **Inv 3d Hermitian** are as follows:

$$f_i = \frac{a}{\bar{b}},\ g_i = \frac{b}{\bar{b}},\ i = 2, 3, \ldots, n-1;\ p = \frac{a}{b},\ q = \frac{a}{\bar{b}}.$$

Further, in Item 3 of the procedure, the quantities μ_i are calculated. In our case, we have second-order recurrent relations

$$\bar{b}\mu_i + a\mu_{i+1} + b\mu_{i+2} = 0\,,\ i = n-2, n-3, \ldots, 1,$$

where $\mu_n = 1$, $\mu_{n-1} = -a/\bar{b}$. The solution of this problem is well known (see [2, 6], for instance). As a result of calculations, we get that

$$\mu_i = (-1)^{n-i}\,\frac{\bar{b}}{r}\left[\left(\frac{a+r}{2\bar{b}}\right)^{n+1-i} - \left(\frac{a-r}{2\bar{b}}\right)^{n+1-i}\right],\ i = 1, 2, \ldots, n \quad \text{if } |a| > 2|b| \tag{40}$$

and

$$\mu_i = (-1)^{n-i}\,(n+1-i)\left(\frac{a}{2b}\right)^{i-n},\ i = 1, 2, \ldots, n \quad \text{if } |a| = 2|b|, \tag{41}$$

where

$$r \equiv \sqrt{a^2 - 4|b|^2}.$$

In a similar way, we find expressions for the quantities ν_i determined in Item 4 of the procedure. These quantities satisfy the following second-order recurrent relations:

$$\bar{b}\nu_{i-2} + a\nu_{i-1} + b\nu_i = 0\,,\ i = 3, 4, \ldots, n,$$

where $\nu_1 = 1$, $\nu_2 = -a/b$. Making calculations, we find that

$$\nu_i = (-1)^{i-1}\,\frac{b}{r}\left[\left(\frac{a+r}{2b}\right)^i - \left(\frac{a-r}{2b}\right)^i\right],\ i = 1, 2, \ldots, n \quad \text{if } |a| > 2|b| \tag{42}$$

and

$$\nu_i = (-1)^{i-1}\, i\left(\frac{a}{2b}\right)^{i-1},\ i = 1, 2, \ldots, n \quad \text{if } |a| = 2|b|. \tag{43}$$

In Item 5 of the procedure, the quantity t is calculated. Using the expressions (40) and (41), we get

$$t = (-1)^{n-1}\frac{r}{\bar{b}^2}\left[\left(\frac{a+r}{2\bar{b}}\right)^{n+1} - \left(\frac{a-r}{2\bar{b}}\right)^{n+1}\right]^{-1} \quad \text{if } |a| > 2|b| \tag{44}$$

and

$$t = \frac{(-1)^{n-1}}{n+1}\,\frac{2}{a}\left(\frac{a}{2b}\right)^{n-1} \quad \text{if } |a| = 2|b|. \tag{45}$$

Finally, in Items 6 and 7 of the procedure, the elements x_{ij} of the inverse matrix A^{-1} are calculated. If $|a| > 2|b|$, then we use the formulas (40), (42) and (44). For the values $j = 1, 2, \ldots, n$, we obtain that

$$x_{ij} = \frac{(-1)^{j-i}}{r}\,\frac{\left[\left(\frac{a+r}{2b}\right)^i - \left(\frac{a-r}{2b}\right)^i\right]\left[\left(\frac{a+r}{2b}\right)^{n+1-j} - \left(\frac{a-r}{2b}\right)^{n+1-j}\right]}{\left[\left(\frac{a+r}{2b}\right)^{n+1} - \left(\frac{a-r}{2b}\right)^{n+1}\right]} \tag{46}$$

if $i = 1, 2, \ldots, j-1$ and

$$x_{ij} = \frac{(-1)^{i-j}}{r}\,\frac{\left[\left(\frac{a+r}{2\bar{b}}\right)^{n+1-i} - \left(\frac{a-r}{2\bar{b}}\right)^{n+1-i}\right]\left[\left(\frac{a+r}{2\bar{b}}\right)^j - \left(\frac{a-r}{2\bar{b}}\right)^j\right]}{\left[\left(\frac{a+r}{2\bar{b}}\right)^{n+1} - \left(\frac{a-r}{2\bar{b}}\right)^{n+1}\right]} \tag{47}$$

if $i = j, j+1, \ldots, n$. As an example, consider the matrix

$$A = \begin{bmatrix} 5 & 2\mathtt{i} & & & \\ -2\mathtt{i} & 5 & 2\mathtt{i} & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -2\mathtt{i} & 5 & 2\mathtt{i} \\ & & & -2\mathtt{i} & 5 \end{bmatrix}.$$

According to the expressions (46) and (47) we find that

$$x_{ij} = \begin{cases} \dfrac{(2^i - 2^{-i})(2^{n+1-j} - 2^{-n-1+j})}{3(2^{n+1} - 2^{-n-1})}\,\mathtt{i}^{i-j}, & i = 1, 2, \ldots, j-1, \\[2ex] \dfrac{(2^{n+1-i} - 2^{-n-1+i})(2^j - 2^{-j})}{3(2^{n+1} - 2^{-n-1})}\,\mathtt{i}^{i-j}, & i = j, j+1, \ldots, n, \end{cases}$$

where the symbol $\mathtt{i}$ stands for the imaginary unit.

Now consider the case $|a| = 2|b|$. For the values $j = 1, 2, \ldots, n$, using the formulas (41), (43) and (45), we find that

$$x_{ij} = \begin{cases} (-1)^{j-i} \dfrac{(n+1-j)i}{n+1} \dfrac{2}{a} \left(\dfrac{a}{2b}\right)^{i-1} \left(\dfrac{a}{2\bar{b}}\right)^{j-1}, & i = 1, 2, \ldots, j-1, \\ (-1)^{i-j} \dfrac{(n+1-i)j}{n+1} \dfrac{2}{a} \left(\dfrac{a}{2b}\right)^{i-1} \left(\dfrac{a}{2\bar{b}}\right)^{j-1}, & i = j, j+1, \ldots, n. \end{cases} \tag{48}$$

For the matrix

$$A = \begin{bmatrix} 2 & \mathtt{i} & & & \\ -\mathtt{i} & 2 & \mathtt{i} & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -\mathtt{i} & 2 & \mathtt{i} \\ & & & -\mathtt{i} & 2 \end{bmatrix},$$

the expressions (48) take the following form:

$$x_{ij} = \begin{cases} (-1)^j \dfrac{(n-j+1)i}{n+1} \mathtt{i}^{i+j}, & i = 1, 2, \ldots, j-1, \\ (-1)^j \dfrac{(n-i+1)j}{n+1} \mathtt{i}^{i+j}, & i = j, j+1, \ldots, n, \end{cases} \qquad j = 1, 2, \ldots, n.$$

5. Conclusion

In this paper, we have constructed the computational procedure **Inv 3d Hermitian** for inversion of tridiagonal Hermitian matrices. This procedure can be used as a numerical algorithm with an optimal number of arithmetic operations (see the comment on the procedure at the end of Section 3). In certain cases, the procedure can also be used to derive closed-form expressions for the elements of inverse matrices. In this regard, Toeplitz tridiagonal Hermitian matrices in Section 4 were considered.

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Եռանկյունագծային հերմիտյան մատրիցների անալիտիկ հակադարձում

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Ամփոփում

Հոդվածում տրվում է եռանկյունագծային հերմիտյան մատրիցների հակադարձման ալգորիթմը, որի թվային իրականացումը պահանջում է օպտիմալ թվով թվաբանական գործողություններ: Հաշվողական պրոցեդուրան իրենից ներկայացնում է հակադարձ մատրիցի տարրերի հաշվմանը հանգեցնող անդրադարձ առնչությունների հաջորդականության: Հատուկ տիպի մատրիցների համար և, մասնավորապես, տյոպլիցյան եռանկյունագծային հերմիտյան մատրիցների համար, ստացված առնչությունները հանգեցնում են հակադարձ մատրիցի տարրերի համար բացահայտ բանաձևերի:

Բանալի բառեր՝ հակադարձ մատրից, եռանկյունագծային մատրից, հերմիտյան մատրից, տյոպլիցյան մատրից:

Аналитическое обращение трехдиагональных изображени

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Аннотация

В статье дается алгоритм обращения трехдиагональных эрмитовых матриц, численная реализация которого осуществляется за оптимальное число арифметических операций. Вычислительная процедура представляет собой

последовательность рекуррентных соотношений, приводящих к вычислению элементов обратной матрицы. Для матриц специального типа и, в частности, для тёплицевых трехдиагональных эрмитовых матриц, полученные соотношения приводят к явным формулам для элементов обратной матрицы.

Ключевые слова: обратная матрица, трехдиагональная матрица, эрмитова матрица, тёплицева матрица.