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# Complete Caps in Affine Geometry $AG(n, 3)$

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**Abstract**

We consider the problem of constructing complete caps in affine geometry $AG(n, 3)$ of dimension $n$ over the field $F_3$ of order three. We will take the elements of $F_3$ to be 0, 1 and 2. A cap is a set of points, no three of which are collinear. Using the concept of $P_n$ −set, we give two new methods for constructing complete caps in affine geometry $AG(n, 3)$. These methods lead to some new upper and lower bounds on the possible minimal and maximal cardinality of complete caps in affine geometry $AG(n, 3)$.

**Keywords:** Affine geometry, Projective geometry, Cap, Complete cap.
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## 1. Introduction

A cap in an affine geometry $AG(n, q)$ or in a projective geometry $PG(n, q)$ over a finite field $F_q$ is a set of points no three of which are collinear. A cap is called complete when it cannot be extended to a large cap. The central problem in the theory of caps is to find the maximal and minimal sizes of caps in the affine geometry $AG(n, q)$ or in the projective geometry $PG(n, q)$. In this paper, $s_{n,q}$ and $s'_{n,q}$ denote the size of the largest caps in $AG(n, q)$ and $PG(n, q)$, respectively. Presently, only the following exact values are known: $s_{n,2} = s'_{n,2} = 2^n$, $s_{2,q} = s'_{2,q} = q + 1$ if $q$ is odd, $s_{2,q} = s'_{2,q} = q + 2$ if $q$ is even, and $s'_{3,q} = q^2 + 1, s_{3,q} = q^2$ [1, 2]. Aside from these general results, the precise values are known only in the following cases: $s_{4,3} = s'_{4,3} = 20$ [3], $s'_{5,3} = 56$ [4], $s_{5,3} = 45$ [5], $s'_{4,4} = 41$ [6], $s_{6,3} = 112$ [7]. In the other cases, only lower and upper bounds on the sizes of caps in $AG(n, q)$ and $PG(n, q)$ are known. Finding the exact value for $s_{n,q}$ and $s'_{n,q}$ in the general case seems to be a very hard problem [8–10]. The only complete cap in $AG(n, 2)$ is the whole $AG(n, 2)$. The trivial lower bound for the size of the smallest complete cap in $AG(n,q)$ is $\sqrt{2}q^{\frac{n-1}{2}}$. For even $q$ there exist complete caps in geometry $AG(n,q)$ with less than $q^{\frac{n}{2}}$ points. But for odd $q$ complete caps in $AG(n,q)$ with less than $q^{\frac{n}{2}}$ points are known to exist [11, 12] only for $n = 0(mod\ 4)$, $n = 2(mod\ 4)$. For more information about complete caps, for small values $n$ and $q$, we refer the reader to [10–13]. Note that the problem of determining the minimum size of a complete cap in a given geometry is of particular interest in Coding theory. Using the concept of a $P_n$-set, which was introduced by the author in 2015 [14], we give two new methods for constructing complete caps in the affine geometry $AG(n,3)$. These methods yield some new upper and lower bounds on the possible minimal and maximal sizes of complete caps in the affine geometry $AG(n,3)$.

## 2. Main Results

We will write the points of $AG(n,q)$ in the following way: $\boldsymbol{x} = (x_1, \cdots, x_n)$, and let us denote by $\boldsymbol{0} = (0, \cdots, 0)$ the origin point of the geometry $AG(n,3)$. It is easy to check that if $\boldsymbol{S}$ is a cap in $AG(n,3)$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \boldsymbol{0}\ (mod\ 3)$ for every triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \boldsymbol{S}$. Let's denote by $B_n = \{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) | \alpha_i = 1, 2\}$ and by $P_n$ the set of points of $AG(n,3)$ satisfying the following two conditions:

i) for any two distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P_n$, there exists $i\ (1 \leq i \leq n)$ such that $\alpha_i = \beta_i = 0$,

ii) for any triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n$, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \boldsymbol{0} (mod\ 3)$.

We say $P_n$ to be complete when it cannot be extended to a larger one. We will define the concatenation of the points of the sets in the following way. Let $A \subset AG(n,3)$ and $B \subset AG(m,3)$. We form a new set $AB \subset AG(n+m,3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{n+m})$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_n) \in A$ and $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, \cdots, \alpha_{n+m}) \in B$. In a similar way, one can define the concatenation of the points for any number of sets.

**Claim 1**. Note that if $x, y, z \in F_3$, then $x + y + z = 0\ (mod\ 3)$ if and only if $x = y = z$ or they are pairwise distinct numbers.
The following two theorems, which we need, are proven in [16, 17].

**Theorem 1**: *The following recurrence relation* $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, *with initial sets* $P_1 = \{(0)\}$, $P_2 = \{(0,1),(0,2)\}$ *and* $n = \sum_{j=1}^{3} n_j$, *yields a complete* $P_n$ *set.*

Having the sets $P_{n_1}$, $P_{n_2}$, $P_{n_3}$, $P_{n_4}$, $P_{n_5}$, $P_{n_6}$ and $B_{n_1}$, $B_{n_2}$, $B_{n_3}$, $B_{n_4}$, $B_{n_5}$, $B_{n_6}$, let us form the following ten sets, by concatenation of the points of the sets.

$$A_1 = P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6}, \qquad A_2 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6},$$
$$A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}, \qquad A_4 = B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6},$$
$$A_5 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6}, \qquad A_6 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6},$$
$$A_7 = B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}, \qquad A_8 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6},$$
$$A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}, \qquad A_{10} = P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}.$$

**Theorem 2:** *The following recurrence relation* $P_n = \bigcup_{i=1}^{10} A_i$, *with initial sets* $P_1 = \{(0)\}$, $P_2 = \{(0,\ 1),(0,2)\}$ *and* $n = \sum_{i=1}^{6} n_i$ *yields a complete* $P_n$ *set.*

**Claim 2**. Note that from the construction of $P_n$ in both theorems it follows that for every i ($1 \leq i \leq n$), if the point $\boldsymbol{p} = (p_1, \ldots, p_i, \ldots, p_n) \in P_n$ and $p_i \neq 0$, then, also, the point $\boldsymbol{p}' = (p_1, \ldots, p_i^{-1}, \ldots, p_n) \in P_n$, where $p_i^{-1}$ is the additive inverse of $p_i$ in the field $F_3$.

The following two main theorems without proofs were first presented at CSIT 2015 in a weak form [14], that they yield caps. But at CSIT 2017 they were presented with a strong conclusion that they yield complete caps [15]. In this paper, we give their complete proofs.

**Theorem 3:** *If* $P_n$ *and* $P_m$ *are constructed either by Theorem 1 or by Theorem 2, then for the given natural numbers n and m, the set* $S = P_n B_m \cup B_n P_m$ *is a complete cap in the geometry* $AG(n+m,3)$.

**Proof**. First of all we will prove that the set $S = P_n B_m \cup B_n P_m$ is a cap. Suppose, to the contrary, that $S$ is not a cap. Then there is a triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in S$, such that $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} = \mathbf{0}(mod\ 3)$. Let's represent the points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}$, $\boldsymbol{\beta} = \boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}$ and $\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(1)}\boldsymbol{\gamma}^{(2)}$, respectively, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, \cdots, \alpha_{n+m})$, $\boldsymbol{\beta}^{(1)} = (\beta_1, \cdots, \beta_n)$, $\boldsymbol{\beta}^{(2)} = (\beta_{n+1}, \cdots, \beta_{n+m})$, $\boldsymbol{\gamma}^{(1)} = (\gamma_1, \cdots, \gamma_n)$ and $\boldsymbol{\gamma}^{(2)} = (\gamma_{n+1}, \cdots, \gamma_{n+m})$. Thus, we obtain $\boldsymbol{\alpha}^{(1)} + \boldsymbol{\beta}^{(1)} + \boldsymbol{\gamma}^{(1)} = \mathbf{0}(mod\ 3)$ and $\boldsymbol{\alpha}^{(2)} + \boldsymbol{\beta}^{(2)} + \boldsymbol{\gamma}^{(2)} = \mathbf{0}(mod\ 3)$. If all three points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in P_n B_m$, then it follows that $\boldsymbol{\alpha}^{(1)},\boldsymbol{\beta}^{(1)},\boldsymbol{\gamma}^{(1)} \in P_n$ and $\boldsymbol{\alpha}^{(2)},\boldsymbol{\beta}^{(2)},\boldsymbol{\gamma}^{(2)} \in B_m$. The definition of the set $P_n$ implies that $\boldsymbol{\alpha}^{(1)} = \boldsymbol{\beta}^{(1)} = \boldsymbol{\gamma}^{(1)}$ and Claim 1 implies that $\boldsymbol{\alpha}^{(2)} = \boldsymbol{\beta}^{(2)} = \boldsymbol{\gamma}^{(2)}$. Therefore, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$, which contradicts that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are pairwise distinct points. In the same manner, one can prove the case, when all three points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in B_n P_m$, is impossible. Now let us assume that two of these points belong to one set (say $\boldsymbol{\alpha},\boldsymbol{\beta} \in P_n B_m$) and the third point $\boldsymbol{\gamma}$ belongs to the other set (say $\gamma \in B_n P_m$). By definition of $P_n$ there is $i$, $1 \leq i \leq n$, so that $\alpha_i = \beta_i = 0$. But, by definition of $B_n$, $\gamma_i = 1\ or\ 2$. Hence, $\alpha_i + \beta_i + \gamma_i \neq 0(mod\ 3)$, which contradicts that $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} = \mathbf{0}(mod\ 3)$. In a similar way, one can prove the case when two points belong to $B_n P_m$ and the third one belongs to $P_n B_m$ is impossible. Therefore, $S$ is a cap.

We will prove the completeness of S again by contradiction. Suppose that there is a point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m})$, such that $\boldsymbol{\alpha} \notin$ S and S $\cup \{\boldsymbol{\alpha}\}$ is a cap. Let's represent the point $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, \cdots, \alpha_{n+m})$. The following two cases are possible.

**Case** 1. At least one of the sets $P_n \cup \{\boldsymbol{\alpha}^{(1)}\}$ or $P_m \cup \{\boldsymbol{\alpha}^{(2)}\}$ satisfies the condition i). Assume that the set $P_n \cup \{\boldsymbol{\alpha}^{(1)}\}$ satisfies the condition i). If $\boldsymbol{\alpha}^{(1)} \in P_n$, then we can choose two points $\boldsymbol{x}$, $\boldsymbol{y} \in B_m$ in the following way. If $\alpha_i = 0$, then we will assume that $x_i = 1$ and $y_i = 2$, otherwise $x_i = y_i = \alpha_i$, $n+1 \leq i \leq n+m$. Therefore, $\boldsymbol{\alpha}^{(2)} \notin B_m$, since $\boldsymbol{\alpha} \notin$ S and $\boldsymbol{\alpha}^{(1)} \in P_n$. Hence, $\boldsymbol{\alpha}^{(2)}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are pairwise distinct points. It is not difficult to see that $\boldsymbol{\alpha}^{(1)}\boldsymbol{x}$, $\boldsymbol{\alpha}^{(1)}\boldsymbol{y} \in P_n B_m$. Claim 1 implies that $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}+\boldsymbol{\alpha}^{(1)}\boldsymbol{x}+\boldsymbol{\alpha}^{(1)}\boldsymbol{y}=\mathbf{0}(mod\ 3)$, which contradicts the assumption that $S\cup\{\boldsymbol{\alpha}\}$ is a cap. If $\boldsymbol{\alpha}^{(1)}\notin P_n$, then the completeness of the $P_n$ implies that there are two distinct points $\boldsymbol{\beta},\boldsymbol{\gamma}\in P_n$, such that $\boldsymbol{\alpha}^{(1)}+\boldsymbol{\beta}+\boldsymbol{\gamma}=\mathbf{0}(mod\ 3)$. Now, as described above, we will choose two points $\boldsymbol{x},\boldsymbol{y}\in B_m$ in the following way. If $\alpha_i=0$, then we will take $x_i=1$ and $y_i=2$, otherwise $x_i=y_i=\alpha_i$, $n+1\le i\le n+m$. The choice of the points $\boldsymbol{x},\boldsymbol{y}$ implies that $\boldsymbol{x},\boldsymbol{y}\in B_m$ and $\boldsymbol{\alpha}^{(2)}+\boldsymbol{x}+\boldsymbol{y}=\mathbf{0}(mod\ 3)$. Therefore, $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}+\boldsymbol{\beta}\boldsymbol{x}+\boldsymbol{\gamma}\boldsymbol{y}=\mathbf{0}(mod3)$, which contradicts the assumption that $S\cup\{\boldsymbol{\alpha}\}$ is a cap. Similarly, one can prove the case, when the set $P_m\cup\{\boldsymbol{\alpha}^{(2)}\}$ satisfies the condition i), is impossible.

**Case 2**. Both sets $P_n\cup\{\boldsymbol{\alpha}^{(1)}\}$ and $P_m\cup\{\boldsymbol{\alpha}^{(2)}\}$ do not satisfy the condition i). Therefore, the condition i) for the set $P_n\cup\{\boldsymbol{\alpha}^{(1)}\}$ follows that there is a point $\boldsymbol{\beta}\in P_n$, such that if $\alpha_i=0$, then $\beta_i\neq 0$ and if $\beta_i=0$, then $\alpha_i\neq 0$, $1\le i\le n$. We will choose the point $\boldsymbol{x}\in B_n$ in the following way. If $\alpha_i=0$, then $x_i=\beta_i^{-1}$ and if $\beta_i=0$, then $x_i=\alpha_i^{-1}$, otherwise, using Claim 2, we can assume that $x_i=\beta_i=\alpha_i$, $1\le i\le n$. By the same reason, the condition i) for the set $P_m\cup\{\boldsymbol{\alpha}^{(2)}\}$ implies that there is a point $\boldsymbol{\gamma}\in P_m$, so that if $\alpha_i=0$, then $\gamma_i\neq 0$ and if $\gamma_i=0$, then $\alpha_i\neq 0, n+1\le i\le n+m$. In the same manner, we will choose the point $\boldsymbol{y}\in B_m$. If $\alpha_i=0$, then $y_i=\gamma_i^{-1}$ and if $\gamma_i=0$, then $y_i=\alpha_i^{-1}$, otherwise, by Claim 2, we can assume that $y_i=\gamma_i=\alpha_i$, $n+1\le i\le n+m$). It is obvious that $\boldsymbol{\beta}\boldsymbol{y}\in P_nB_m$ and $\boldsymbol{x}\boldsymbol{\gamma}\in B_nP_m$. The choice of the points $\boldsymbol{x},\boldsymbol{y}$ implies that $\boldsymbol{\alpha}^{(1)}+\boldsymbol{\beta}+\boldsymbol{x}=\mathbf{0}(mod\ 3)$ and $\boldsymbol{\alpha}^{(2)}+\boldsymbol{\gamma}+\boldsymbol{y}=\mathbf{0}(mod\ 3)$. Therefore, $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}+\boldsymbol{\beta}\boldsymbol{y}+\boldsymbol{x}\boldsymbol{\gamma}=\mathbf{0}(mod\ 3)$, which again contradicts the assumption that $S\cup\{\boldsymbol{\alpha}\}$ is a cap.

□

**Corollary 1:** *For the given natural numbers n and m,* $s_{n+m,3}\ge|P_n||B_m|+|B_n||P_m|$.

**Corollary 2:** *For every natural number n,* $s_{n+1,3}\ge 2|P_n|+|B_n|$.

**Theorem 4:** *If $P_n$ and $P_m$ are constructed by Theorem 1 or by Theorem 2, then for the given natural numbers n and m, $S=P_nP_m\{0\}\cup P_nB_m\{1\}\cup B_nP_m\{1\}\cup B_{n+m}\{2\}$ is a complete cap in the geometry $AG(n+m+1,3)$.*

**Proof.** First we will prove that the set $S=P_nP_m\{0\}\cup P_nB_m\{1\}+B_nP_m\{1\}+B_{n+m}\{2\}$ is a cap by contradiction. Assume that there are three distinct points $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n,\alpha_{n+1},\dots,\alpha_{n+m},\alpha_{n+m+1})$, $\boldsymbol{\beta}=(\beta_1,\dots,\beta_n,\beta_{n+1},\dots,\beta_{n+m},\beta_{n+m+1})$, $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_n,\gamma_{n+1},\dots,\gamma_{n+m},\gamma_{n+m+1})\in S$, such that $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}=\mathbf{0}(mod\ 3)$. Therefore, $\boldsymbol{\alpha}^{(1)}+\boldsymbol{\beta}^{(1)}+\boldsymbol{\gamma}^{(1)}=\mathbf{0}(mod\ 3)$, $\boldsymbol{\alpha}^{(2)}+\boldsymbol{\beta}^{(2)}+\boldsymbol{\gamma}^{(2)}=\mathbf{0}(mod\ 3)$ and $\alpha_{n+m+1}+\beta_{n+m+1}+\gamma_{n+m+1}=\mathbf{0}(mod\ 3)$, where $\boldsymbol{\alpha}^{(1)}=(\alpha_1,\cdots,\alpha_n)$, $\boldsymbol{\alpha}^{(2)}=(\alpha_{n+1},\cdots,\alpha_{n+m})$, $\boldsymbol{\beta}^{(1)}=(\beta_1,\cdots,\beta_n)$, $\boldsymbol{\beta}^{(2)}=(\beta_{n+1},\cdots,\beta_{n+m})$, $\boldsymbol{\gamma}^{(1)}=(\gamma_1,\cdots,\gamma_n)$ and $\boldsymbol{\gamma}^{(2)}=(\gamma_{n+1},\cdots,\gamma_{n+m})$. Claim 1 implies that $\alpha_{n+m+1}=\beta_{n+m+1}=\gamma_{n+m+1}$ or $\alpha_{n+m+1}$, $\beta_{n+m+1}$, and $\gamma_{n+m+1}$ are pairwise distinct numbers. Hence, the following four cases are possible.

**Case 1.** $\alpha_{n+m+1}=\beta_{n+m+1}=\gamma_{n+m+1}=0$. Therefore, $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in P_nP_m\{0\}$, $\boldsymbol{\alpha}^{(1)},\boldsymbol{\beta}^{(1)},\boldsymbol{\gamma}^{(1)}\in P_n$ and $\boldsymbol{\alpha}^{(2)},\boldsymbol{\beta}^{(2)},\boldsymbol{\gamma}^{(2)}\in P_m$. From the definition of $P_n$ and $P_m$ and the two relations $\boldsymbol{\alpha}^{(1)}+\boldsymbol{\beta}^{(1)}+$

$\boldsymbol{\gamma}^{(1)} = 0(mod\ 3)$, $\boldsymbol{\alpha}^{(2)} + \boldsymbol{\beta}^{(2)} + \boldsymbol{\gamma}^{(2)} = \mathbf{0}(mod\ 3)\ it$ follows that $\boldsymbol{\alpha}^{(1)} = \boldsymbol{\beta}^{(1)} = \boldsymbol{\gamma}^{(1)}$ and $\boldsymbol{\alpha}^{(2)} = \boldsymbol{\beta}^{(2)} = \boldsymbol{\gamma}^{(2)}$. Hence, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$, which contradicts the assumption that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise distinct points.

**Case 2.** $\alpha_{n+m+1} = \beta_{n+m+1} = \gamma_{n+m+1}$=1. Assume that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n B_m\{1\}$. Then $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\beta}^{(1)}, \boldsymbol{\gamma}^{(1)} \in P_n$ and $\boldsymbol{\alpha}^{(2)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\gamma}^{(2)} \in B_m$. The definition of $P_n$ implies that $\boldsymbol{\alpha}^{(1)} = \boldsymbol{\beta}^{(1)} = \boldsymbol{\gamma}^{(1)}$**,** since $\boldsymbol{\alpha}^{(1)} + \boldsymbol{\beta}^{(1)} + \boldsymbol{\gamma}^{(1)} = \mathbf{0}(mod\ 3)$**.** Because $\boldsymbol{\alpha}^{(2)} + \boldsymbol{\beta}^{(2)} + \boldsymbol{\gamma}^{(2)} = \mathbf{0}(mod\ 3)$, Claim 1 implies that $\boldsymbol{\alpha}^{(2)} = \boldsymbol{\beta}^{(2)} = \boldsymbol{\gamma}^{(2)}$**.** Therefore, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$**,** which, again contradicts the assumption that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise distinct points. Similarly, one can prove that the case is impossible, when $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in B_n P_m\{1\}$. Therefore, two points, say $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P_n B_m\{1\}$ and $\boldsymbol{\gamma} \in B_n P_m\{1\}$. The definition of $P_n$ implies that there is $i$, such that $\alpha_i = \beta_i = 0$, $1 \le i \le n$, . But by the definition of $B_n$, $\gamma_i = 1\ or\ 2$. Hence, $\alpha_i + \beta_i + \gamma_i \ne 0(mod\ 3)$, which contradicts that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} = \mathbf{0}(mod\ 3)$. In a similar manner, one can prove that the case is impossible, when two points from $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ belong to $B_n P_m$ and the third one belongs to $P_n B_m$. Therefore, $S$ is a cap.

**Case 3**. $\alpha_{n+m+1} = \beta_{n+m+1} = \gamma_{n+m+1} = 2$. Therefore $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in B_{n+m}\{2\}$. Hence, $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}$, $\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}, \boldsymbol{\gamma}^{(1)}\boldsymbol{\gamma}^{(2)} \in B_{n+m}$ and $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)} + \boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)} + \boldsymbol{\gamma}^{(1)}\boldsymbol{\gamma}^{(2)} = \mathbf{0}(mod\ 3)$. Claim 1 implies that $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)} = \boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)} = \boldsymbol{\gamma}^{(1)}\boldsymbol{\gamma}^{(2)}$. This yields $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$**,** which, again contradicts the assumption that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise distinct points.

**Case** $\alpha_{n+m+1}$, $\beta_{n+m+1}$ and $\gamma_{n+m+1}$ are pairwise distinct numbers. Without loss of generality, let us assume that $\alpha_{n+m+1} = 0$, $\beta_{n+m+1} = 1$ and $\gamma_{n+m+1} = 2$. Therefore, $\boldsymbol{\alpha} \in P_n P_m\{0\}, \boldsymbol{\beta} \in P_n B_m\{1\}$ or $\boldsymbol{\beta} \in B_n P_m\{1\}$ and $\boldsymbol{\gamma} \in B_{n+m}\{2\}$. If $\boldsymbol{\beta} \in P_n B_m\{1\}$, then $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\beta}^{(1)} \in P_n$. Hence, the definition of $P_n$ implies that there is $i$, such that $\alpha_i = \beta_i = 0, 1 \le i \le n$. But, by the definition of $B_n, \gamma_i = 1$ or 2. Therefore, $\alpha_i + \beta_i + \gamma_i \ne 0(mod\ 3)$, which contradicts that $\boldsymbol{\alpha}^{(1)} + \boldsymbol{\beta}^{(1)} + \boldsymbol{\gamma}^{(1)} = \mathbf{0}(mod\ 3)$. The last relation, in turn, implies that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \ne \mathbf{0}(mod\ 3)$. In a similar manner, one can prove the case when $\boldsymbol{\beta} \in B_n P_m\{1\}$ is impossible. Hence, $S$ is a cap.

Now we will prove the completeness of $S$ also by contradiction. Let us assume that there is a point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m}, \alpha_{n+m+1})$, such that $\boldsymbol{\alpha} \notin S$ and $S \cup \{\boldsymbol{\alpha}\}$ is a cap. The following three cases are possible.

**Case** $\alpha_{n+m+1} = 2$. Since $\boldsymbol{\alpha} \notin S$, we have $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m}) \notin B_{n+m}$. We can choose two points $\boldsymbol{x}, \boldsymbol{y} \in B_{n+m}\{2\}$, such that, if $\alpha_i = 0$ then $x_i = 2$ and $y_i = 1$, otherwise $x_i = y_i = \alpha_i$, $1 \le i \le n+m$. It is obvious that $\boldsymbol{x}\{2\}$**,** $\boldsymbol{y}\{2\} \in B_{n+m}\{2\}$ and $\boldsymbol{\alpha}, \boldsymbol{x}\{2\}$, $\boldsymbol{y}\{2\}$ are pairwise distinct points. Claim 1 implies that $\boldsymbol{x}\{2\} + \boldsymbol{y}\{2\} + \boldsymbol{\alpha} = \mathbf{0}(mod\ 3)$, which contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap.

**Case** $\alpha_{n+m+1} = 1$. Let's represent the point $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\{1\}$**,** where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_n)$ and $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, \cdots, \alpha_{n+m})$. Assume that at least one of the sets $P_n \cup \{\boldsymbol{\alpha}^{(1)}\}$ or $P_m \cup \{\boldsymbol{\alpha}^{(2)}\}$ satisfies the condition i), say $P_n \cup \{\boldsymbol{\alpha}^{(1)}\}$**.** First, suppose that $\boldsymbol{\alpha}^{(1)} \notin P_n$. Then the completeness of the set $P_n$ follows that there are two points $\boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n$, such that $\boldsymbol{\beta} + \boldsymbol{\gamma} + \boldsymbol{\alpha}^{(1)} = \mathbf{0}(mod\ 3)$**.** We will choose two points $\boldsymbol{x}, \boldsymbol{y} \in B_m$ in the following way. If $\alpha_i = 0$, then $x_i = 1$ and $y_i = 2$, otherwise $x_i = y_i = \alpha_i$, $n+1 \le i \le n+m$. From the choice of the points $\boldsymbol{x}, \boldsymbol{y}$ it follows that $\boldsymbol{x}, \boldsymbol{y} \in B_m$ and $\boldsymbol{\alpha^{(2)}} + \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}(mod\ 3)$. Therefore, $\boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{1\} + \boldsymbol{\beta}\boldsymbol{x}\{1\} + \boldsymbol{\gamma}\boldsymbol{y}\{1\} = \boldsymbol{0}(mod\ 3)$, which contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap. Otherwise, if $\boldsymbol{\alpha^{(1)}} \in P_n$, then $\boldsymbol{\alpha^{(2)}} \notin B_m$, because $\boldsymbol{\alpha} \notin S$. Then it is easy to see that $\boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{1\} + \boldsymbol{\alpha^{(1)}}\boldsymbol{x}\{1\} + \boldsymbol{\alpha^{(1)}}\boldsymbol{y}\{1\} = \boldsymbol{0}(mod\ 3)$, which, again contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap. Similarly, one can prove the case, when the set $P_m \cup \{\boldsymbol{\alpha^{(2)}}\}$ satisfies the condition i) is impossible. Therefore, both sets $P_n \cup \{\boldsymbol{\alpha^{(1)}}\}$ and $P_m \cup \{\boldsymbol{\alpha^{(2)}}\}$ do not satisfy the condition i). Hence, there is a point $\boldsymbol{\beta} \in P_n$, (respectively, $\boldsymbol{\gamma} \in P_m$), such that if $\alpha_i = 0$, then $\beta_i \neq 0$ and if $\beta_i = 0$, then $\alpha_i \neq 0, 1 \le i \le n$ (respectively, if $\alpha_i = 0$, then $\gamma_i \neq 0$ and if $\gamma_i = 0$, then $\alpha_i \neq 0, n+1 \le i \le n+m$ ). First, let's choose the point $\boldsymbol{x} \in B_n$ in the following way. If $\alpha_i = 0$, then $x_i = \beta_i^{-1}$ and if $\beta_i = 0$, then $x_i = \alpha_i^{-1}$, otherwise, by Claim 2, we can assume that $x_i = \beta_i = \alpha_i$, $1 \le i \le n$. In the same manner, we will choose the point $\boldsymbol{y} \in B_m$. If $\alpha_i = 0$, then $y_i = \gamma_i^{-1}$ and if $\gamma_i = 0$, then $y_i = \alpha_i^{-1}$, otherwise, using Claim 2, we can assume that $y_i = \gamma_i = \alpha_i$, $n+1 \le i \le n+m$). The choice of the points $\boldsymbol{x}$ and $\boldsymbol{y}$ implies that $\boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{1\} + \boldsymbol{\beta}\boldsymbol{y}\{1\} + \boldsymbol{x}\boldsymbol{\gamma}\{1\} = \boldsymbol{0}(mod\ 3)$, which again contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap.

**Case** $\alpha_{n+m+1} = 0$. Assume that at least one of the sets $P_n \cup \{\boldsymbol{\alpha^{(1)}}\}$ or $P_m \cup \{\boldsymbol{\alpha^{(2)}}\}$ does not satisfy the condition i), say the set $P_n \cup \{\boldsymbol{\alpha^{(1)}}\}$**.** Therefore, the condition i) implies that there is a point $\boldsymbol{\beta} \in P_n$, such that, if $\alpha_i = 0$, then $\beta_i \neq 0$ and if $\beta_i = 0$, then $\alpha_i \neq 0, 1 \le i \le n$. We will choose the points $\boldsymbol{z^{(1)}} \in B_n$ and $\boldsymbol{z^{(2)}}$, $\boldsymbol{y} \in B_m$ in the following way. First let's choose $\boldsymbol{z^{(1)}}$**.** If $\alpha_i = 0$, then $z_i = \beta_i^{-1}$ and if $\beta_i = 0$, then $z_i = \alpha_i^{-1}$, otherwise, using Claim 2, we will assume that $z_i = \beta_i = \alpha_i$, $1 \le i \le n$. Now we will choose the points $\boldsymbol{z^{(2)}}$, $\boldsymbol{y} \in B_m$ in the following way. If $\alpha_i = 0$, then we will assume that $z_i = 1$ and $y_i = 2$, otherwise $z_i = y_i = \alpha_i, n+1 \le i \le n+m$. It is easy to see that $\boldsymbol{\beta}\boldsymbol{y}\{1\} \in P_nB_m\{1\}, \boldsymbol{z^{(1)}}\boldsymbol{z^{(2)}}\{2\} \in B_{n+m}\{2\}$. The choice of the points $\boldsymbol{z^{(1)}}, \boldsymbol{z^{(2)}}$ and $\boldsymbol{y}$ imply that $\boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{0\} + \boldsymbol{\beta}\boldsymbol{y}\{1\} + \boldsymbol{z^{(1)}}\boldsymbol{z^{(2)}}\{2\} = \boldsymbol{0}(mod\ 3)$, which contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap. Similarly, one can prove the case is impossible, when the set $P_m \cup \{\boldsymbol{\alpha^{(2)}}\}$ does not satisfy the condition i). Therefore, both sets $P_n \cup \{\boldsymbol{\alpha^{(1)}}\}$ and $P_m \cup \{\boldsymbol{\alpha^{(2)}}\}$ are satisfying the condition i). Since $\boldsymbol{\alpha} \notin S$, therefore either $\boldsymbol{\alpha^{(1)}} \notin P_n$ or $\boldsymbol{\alpha^{(2)}} \notin P_m$. If $\boldsymbol{\alpha^{(1)}} \notin P_n$ and $\boldsymbol{\alpha^{(2)}} \in P_m$, then the completeness of $P_n$ follows that there are two points $\boldsymbol{x}, \boldsymbol{y} \in P_n$, so that $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{\alpha^{(1)}} = \boldsymbol{0}(mod\ 3)$. Since $\boldsymbol{x}, \boldsymbol{y} \in P_n$ and $\boldsymbol{\alpha^{(2)}} \in P_m$, we have $\boldsymbol{x}\boldsymbol{\alpha^{(2)}}$, $\boldsymbol{y}\boldsymbol{\alpha^{(2)}} \in P_nP_m$ and $\boldsymbol{x}\boldsymbol{\alpha^{(2)}}\{0\} + \boldsymbol{y}\boldsymbol{\alpha^{(2)}}\{0\} + \boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{0\} = \boldsymbol{0}(mod\ 3)$, which contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap. The case, when $\boldsymbol{\alpha^{(2)}} \notin P_m$ and $\boldsymbol{\alpha^{(1)}} \in P_n$ is analogous to the above described one and therefore is impossible. Hence, $\boldsymbol{\alpha^{(1)}} \notin P_n$ and $\boldsymbol{\alpha^{(2)}} \notin P_m$. Therefore, from the completeness of $P_n$ and $P_m$ it follows that there are points $\boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n$ and $\boldsymbol{\delta}, \boldsymbol{\theta} \in P_m$, so that $\boldsymbol{\beta} + \boldsymbol{\gamma} + \boldsymbol{\alpha^{(1)}} = \boldsymbol{0}(mod\ 3)$ and $\boldsymbol{\delta} + \boldsymbol{\theta} + \boldsymbol{\alpha^{(2)}} = \boldsymbol{0}(mod\ 3)$. The last two relations imply that $\boldsymbol{\alpha^{(1)}}\boldsymbol{\alpha^{(2)}}\{0\} + \boldsymbol{\beta}\boldsymbol{\delta}\{0\} + \boldsymbol{\gamma}\boldsymbol{\theta}\{0\} = \boldsymbol{0}(mod\ 3)$, which contradicts the assumption that $S \cup \{\boldsymbol{\alpha}\}$ is a cap.

□

**Corollary 3:** *For the given natural numbers n and m*, $s_{n+m+1,3} \ge |P_n||P_m| + |P_n||B_m| + |B_n||P_m| + |B_{n+m}|$.

**Corollary 4:** $s_{5,3} \geq 42$.

**Proof.** By definition $P_1$={(0)}. From Theorem 1 it follows that $P_3 = P_{1+1+1} = P_1P_1B_1 \cup P_1B_1P_1 \cup B_1P_1P_1$ $=\{(0,0,1),(0,0,2),(0,1,0),(0,2,0),(1,0,0),(2,0,0)\}$. It is easy to see that $|B_n| = 2^n$. Therefore, $s_{5,3} \geq |P_3||P_1| + |P_3||B_1| + |B_3||P_1| + |B_4|$ $=6 \times 1 + 6 \times 2 + 8 \times 1 + 16 = 42$.

□

## 3. Conclusion

Notice that the cardinality of $P_n$ obtained by Theorem 1 (Theorem 2) [16, 17], essentially depends on the representation of $n$ as the sum of three (six) natural numbers. Presenting the natural numbers as the sum of six natural numbers and applying Theorem 2, for some $n \geq 6$ in some cases, one can obtain larger complete $P_n$ sets than those, which are constructed by Theorem 1. It is easy to check that $|P_1| = 1$, $|P_2| = 2$, and $|P_{1+1+1}| = 6$. $|P_{2+1+1}| = 12$, $|P_{3+1+1}| = 32$, $|P_{1+1+1+1+1+1}| = 80$, $|P_7| = |P_{3+3+1}| = 168$, $|P_8| = |P_{1+1+1+1+1+3}| = 400$, $|P_9| = |P_{3+3+3}| = 864$… It is not difficult to see that the maximal size $|P_n| > 2^n$, if $n > 5$. Therefore, to construct large complete caps it is convenient to use Corollary 2, but for small complete caps one can use Theorem 4.

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# Լրիվ գլխարկներ $AG(n,3)$ աֆինական երկրաչափությունում

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## Ամփոփում

Դիտարկվում է $n$ չափանի $AG(n,3)$ աֆինական երկրաչափությունում լրիվ գլխարկների կառուցման խնդիրը $F_3 = \{0,1,2\}$ դաշտի վրա: Գլխարկը այն կետերի բազմությունն է, որոնցից ոչ մի երեքը համագիծ չեն: Օգտագործելով $P_n$ բազմության հասկացությունը, մշակվել են լրիվ գլխարկների կառուցման երկու նոր մեթոդներ:

**Բանալի բառեր՝** աֆինական երկրաչափություն, պրոյեկտիվ երկրաչափություն, կետեր, գլխարկներ, լրիվ գլխարկներ:

# Полные шапки в аффинной геометрии $AG(n, 3)$

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**Аннотация**

Рассматривается задача построения полных шапок в аффинной геометрии $AG(n, 3)$ размерности n над полем $F_3 = \{0, 1, 2\}$. Шапка — это набор точек, никакие три из которых не коллинеарны. С помощью понятия множества $P_n$, разработаны две новые конструкции построения полных шапок.

**Ключевые слова:** аффинная геометрия, проективная геометрия, точки, шапки, полные шапки.