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On Medial-like Functional Equations

Amir Ehsani

Department of Mathematics
Mahshahr Branch, Islamic Azad University
Mahshahr, Iran.
Mahshahr Branch, Islamic Azad University, Mahshahr, Iran.
a.ehsani@mahshahriau.ac.ir

Let A be a nonempty set, n and m be positive integers and $f : A^n \to A^m$ be an arbitrary function. Then (A, f) is called $[n, m]$-groupoid. The n-ary operations, $f_1, \ldots, f_m$, are defined by the following:

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_m) \Leftrightarrow y_i = f_i(x_1, \ldots, x_n),$$

for every $1 \leq i \leq m$, are called the component operations of f and they are denoted by $f = (f_1, \ldots, f_m)$ [11, 12, 13]. The $[n, m]$-groupoid is proper iff $n, m, |Q| \geq 2$.

The $[n, m]$-groupoid (A, f) is called $[n, m]$-quasigroup (or multiquasigroup [2, 3, 14]) iff for every injection, $\phi : N_n \to N_{n+m}$, where $N_n = \{1, \ldots, n\}$, and every $(a_1, \ldots, a_n) \in Q^n$ there exists a unique $(b_1, \ldots, b_{n+m}) \in Q^{n+m}$ such that:

$$f(b_1, \ldots, b_n) = (b_{n+1}, \ldots, b_{n+m}) \quad \text{and} \quad b_{\phi(i)} = a_i,$$

for $i = 1, \ldots, n$.

It is clear that $Q(f)$ is an $[n, 1]$-quasigroup iff $Q(f)$ is an n-quasigroup [1]. $Q(f)$ is a $[1, m]$-quasigroup iff there exist permutations, $f_1, \ldots, f_m$, of Q such that $f(x) = (f_1(x), \ldots, f_m(x))$. It is also clear that all components of a multiquasigroup are binary quasigroup operations.

If the component operations of the $[n, m]$-quasigroup are binary operations, i.e. $n = 2$, then we say that the $[n, m]$-quasigroup is a binary multiquasigroup.

Let us consider the following hyperidentities [7, 8, 9]:

$$\begin{aligned}
g(f(x,y), f(u,v)) &= f(g(x,u), g(y,v)), && \text{(Mediality)}\\
g(f(x,y), f(u,v)) &= f(g(v,y), g(u,x)), && \text{(Paramediality)}\\
g(f(x,y), f(u,v)) &= g(f(x,u), f(y,v)), && \text{(Co-mediality)}\\
g(f(x,y), f(u,v)) &= g(f(v,y), f(u,x)), && \text{(Co-paramediality)}\\
f(x,x) &= x. && \text{(Idempotency)}
\end{aligned}$$

The binary algebra, (A, F), is called:

- medial, if it satisfies the identity (1.1),
- paramedial, if it satisfies the identity (1.2),

- co-medial, if it satisfies the identity (1.3),
- co-paramedial, if it satisfies the identity (1.4),
- idempotent, if it satisfies the identity (1.5),

for every $f, g \in F$. The binary algebra, (A, F), is called mode, if it is medial and idempotent.

Definition 1 *The binary multiquasigroup (A, f) with $f = (f_1, \ldots, f_m)$ is called:*

- *medial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is medial,*
- *paramedial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is paramedial,*
- *co-medial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is co-medial,*
- *co-paramedial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is co-paramedial,*
- *idempotent, if the binary algebra, $(A, f_1, \ldots, f_m)$, is idempotent,*
- *mode, if the binary algebra, $(A, f_1, \ldots, f_m)$, is a mode.*

The next characterization of binary medial multiquasigroups follows from [6, 10].

Theorem 1 *Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary medial multiquasigroup, then there exists an abelian group, $(Q, +)$, such that:*

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group $(Q, +)$, and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \beta_j\alpha_i, \alpha_i\alpha_j = \alpha_j\alpha_i, \beta_i\beta_j = \beta_j\beta_i$, for $i, j = 1, \ldots, m$. The group, $(Q, +)$, is unique up to isomorphisms. Moreover, if (Q, f) is a mode, then

$$f_i(x, y) = \alpha_i x + \beta_i y,$$

where α_i, β_i are automorphisms of both the group, $(Q, +)$, and of the algebra, $(Q, f_1, \ldots, f_m)$.

In this paper we characterize the binary paramedial, co-medial and co-paramedial multiquasigroups (cf. [4, 5]).

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