On Cycles Through Vertices of Large Semidegree in Digraphs

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Abstract

Let D be a strong digraph on $n = 2m + 1 \geq 5$ vertices. In this paper we show that if D contains a cycle of length $n - 1$, then D has also a cycle which contains all vertices with in-degree and out-degree at least m (unless some extremal cases).
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1. Introduction

The digraph D is hamiltonian if it contains a hamiltonian cycle, i.e. a cycle of length $|V(D)|$. A set S of vertices in a digraph D (an undirected graph G) is said to be cyclable in D (in G) if D (G) contains a cycle through all vertices of S.

There are many well-known conditions which guarantee the cyclability of a set of vertices in an undirected graph. Most of them can be seen as restrictions of hamiltonian conditions to the considered set of vertices (See [4, 5, 15, 16, 18]). However, for general digraphs, relatively few degree conditions are known to guarantee hamiltonisity in digraphs (See [2, 3, 7, 9, 13, 14, 17, 19]). The more general and classical one is the following theorem of M. Meyniel:

Theorem A [13]. *If D is a strong digraph of order $n \geq 2$ and $d(x) + d(y) \geq 2n - 1$ for all pairs of nonadjacent vertices in D, then D is hamiltonian .*

In [8], the first author proved the following:

Theorem B [8]. *Let D be a strong digraph of order $n \geq 3$. If $d(x) + d(y) \geq 2n - 1$ for any two non-adjacent vertices $x, y \in V(D) - \{z_0\}$, where z_0 is some vertex of D, then D is hamiltonian or contains a cycle of length $n - 1$.*
The following result is immediately corollary of Theorem B.

Corollary [8]. *Let D be a strong digraph of order $n \geq 3$. If D has $n - 1$ vertices of degree at least n, then D is hamiltonian or contains a cycle of length $n - 1$.*

A Meyniel set M is a subset of $V(D)$ such that $d(x) + d(y) \geq 2n - 1$ for every pair of vertices x, y in M which are nonadjacent in D. In [4], K. A. Berman and X. Liu improved Theorem B proving the following generalization of the well-known Meyniel's theorem.

Theorem C [4]. *Let D be a digraph of order n. If D is strongly connected, then every Meyniel set M lies in a cycle.*
Theorem C also generalizes the classical theorems A. Ghouila-Houri [11] and D.R. Woodall [19].

The digraph D is S-strongly connected if for any pair x, y of distinct vertices of S there exists a path from x to y and a path from y to x in D (See [12]). H. Li, E. Flandrin and J. Shu [12] proved the following generalization of Theorem C.

Theorem D [12]. *Let D be a digraph of order n and M be a Meyniel set in D. If D is M-strongly connected, then D contains a cycle through all vertices of M.*

C. Thomassen [17] (for $n = 2k + 1$) and the first author [7] (for $n = 2k$) proved the following:

Theorem E [17, 7]. *If D is a digraph of order $n \geq 5$ with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$, then D is hamiltonian (unless some extremal cases which are characterized).*

We put a question to know if this result of C. Thomassen and the first author has a cyclable version.

Let D be a digraph of order $n = 2m + 1$. A Thomassen set T is a subset of $V(D)$ such that $d^+(x) \geq m$ and $d^-(x) \geq m$ for every $x \in T$, we denote the vertices of T by T-vertices. The cycle containing all vertices of T is called a T-cycle.

In this paper we prove the following two theorems which provide some support for the above question.

Theorem 1. *Let D be a strong digraph of order $n = 2m + 1 \geq 3$ and D contains a cycle of length $n - 1$. Then one of the following holds:*

i. D contains a cycle containing all vertices with in-degree and out-degree at least m;

ii. D is isomorphic to digraphs D_5 or D_7 or belongs to the set $L_1 \cup L_2$;

iii. $K_{m,m+1}^* \subseteq D \subseteq [K_m + \overline{K}_{m+1}]^*$;

iv. D contains a cycle $C := x_1 x_2 \ldots x_{2m} x_1$ of length $n - 1$, and if $x \notin V(C)$ and x is not adjacent to the vertices $x_{l_1}, x_{l_2}, \ldots, x_{l_j}$, $j \geq 3$, then $x_{l_i - 1} x, x x_{l_i + 1} \in D$ and $N^+(x) = N^+(x_{l_i})$ and $N^-(x) = N^-(x_{l_i})$ for all $i \in [1, j]$. In particular, $\{x_{l_1}, x_{l_2}, \ldots, x_{l_j}, x\}$ is an independent set of vertices.

Theorem 2. *Let D be a 2-strong digraph of order $n = 2m + 1 \geq 3$. Then any two T-vertices x and y are on a common cycle in D.*

Our proofs are based on the argument of [17, 7].

2.   Terminology and Notations

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraphs) and refer the reader to monograph of Bang-Jensen and Gutin [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The order $|D|$ of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy. For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \rightarrow B$. The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A \rangle$. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). For a cycle $C_k = x_1 x_2 \cdots x_k x_1$, the subscripts considered modulo k, i.e. $x_i = x_s$ for every s and i such that $i \equiv s \pmod k$. If P is a path containing a subpath from x to y we let $P[x, y]$

denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. For an undirected graph G, we denote by G^* symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. K_n (respectively, $K_{p,q}$) denotes the complete graph of order n (respectively, complete bipartite graph with partite sets of cardinalities p and q), and $\overline{K}_n$ denotes the complement of complete undirected graph of order n. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). We denote by $a(x, y)$ the number of arcs between the vertices x and y. In particular, $a(x, y) = 0$ (respectively, $a(x, y) \neq 0$) means that x and y are not adjacent (respectively, are adjacent). For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

3.   Preliminaries

The following well-known simple lemmas are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proofs of our results.

Lemma 1 [10]. *Let D be a digraph on $n \geq 3$ vertices containing a cycle C_m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m + 1$, then D contains a cycle C_k for all $k \in [2, m + 1]$.*

Lemma 2 [6]. *Let D be a digraph on $n \geq 3$ vertices containing a path $P := x_1 x_2 \ldots x_m$, $m \in [2, n - 1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:*

(i) $d(x, P) \geq m + 2$;

(ii) $d(x, P) \geq m + 1$ and $xx_1 \notin D$ or $x_m x_1 \notin D$;

(iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_m x \notin D$

then there is an $i \in [1, m - 1]$ such that $x_i x, xx_{i+1} \in D$, i.e., D contains a path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ of length m (we say that x can be inserted into P or the path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ is extended from P with x).

4.   Proof of Theorem 1

Here we prove only Theorem 1 and for it we need the following definitions.

Definition 1.   *D_7 is a digraph (see [1, 17]) with the vertex set $V(D_7) = \{x_1, x_2, x_3, x_4, x_5, x, y\}$ such that $N^+(x_1) = \{x_2, x_5, y\}$, $N^+(x_2) = \{x_3, x_4, y\}$, $N^+(x_3) = \{x_2, x_4, x\}$, $N^+(x_4) = \{x_3, x_5, x\}$, $N^+(x_5) = \{x_1, x, y\}$, $N^+(x) = \{x_1, x_2, x_3\}$ and $N^+(y) = \{x_1, x_4, x_5\}$.*

Definition 2. *D_5 is a digraph (see [1, 17]) with the vertex set $V(D_5) = \{x_1, x_2, x_3, x, y\}$ such that $N^+(x_1) = \{x_2, y\}$, $N^+(x_2) = \{x_3, x\}$, $N^+(x_3) = \{x, y\}$, $N^+(x) = \{x_1, x_2\}$ and $N^+(y) = \{x_1, x_3\}$.*

We denote by L_1 the set of three digraphs obtained from D_5 by adding the arc $x_1 x_3$ or $x_3 x_1$ (or both).

Definition 3. *By L_2 we denote the set of digraphs D with the vertex set $V(D) = \{x_1, x_2, \ldots, x_{2m}, x\}$ and with the following properties:*

i. D contains a cycle $x_1 x_2 \ldots x_{2m} x_1$ of length $2m$ and the vertices x and x_{2m} are not adjacent;

ii. $N^+(x) = N^+(x_{2m}) = \{x_1, x_2, \ldots, x_m\}$ and $N^-(x) = N^-(x_{2m}) = \{x_m, x_{m+1}, \ldots, x_{2m-1}\}$;

iii. $A(\{x_1, x_2, \ldots, x_{m-1}\} \rightarrow \{x_{m+1}, x_{m+2}, \ldots, x_{2m-1}\}) = \emptyset$, the induced subdigraphs $\langle\{x_1, x_2, \ldots, x_m\}\rangle$ and $\langle\{x_m, x_{m+1}, \ldots, x_{2m-1}\}\rangle$ are arbitrary and one may add any number of arcs that go from $\{x_{m+1}, x_{m+2}, \ldots, x_{2m-1}\}$ to $\{x_1, x_2, \ldots, x_m\}$. (Note that the digraphs from L_2 are not 2-strong and x, x_{2m} are T-vertices which are not in the common cycle.

Proof of Theorem 1. The proof is by contradiction. Suppose that Theorem 2 is false, in particular, D is not hamiltonian. Let $C := x_1 x_2 \ldots x_{n-1} x_1$ be an arbitrary cycle of length $n-1$ in D and let the vertex x is not containing this cycle C. In further, by H we denote a hamiltonian cycle in D.

Then x is a T-vertex. Since C is a longest cycle, using Lemmas 1 and 2, we obtain the following claim:

Claim 1. (i). $d(x) = n - 1$ and there is a vertex x_l, $l \in [1, n-1]$ which is not adjacent to x.

(ii). If $x_i x \notin D$, then $x x_{i+1} \in D$ and if $x x_i \notin D$, then $x_{i-1} x \in D$, where $i \in [1, n-1]$.

(iii). If the vertices x and x_i are not adjacent, then $x_{i-1} x$, $x x_{i+1} \in D$ and $d(x_i) = n - 1$.

$\square$

By Claim 1(i), without loss of generality, we may assume that the vertices x and x_{n-1} are not adjacent. For convenience, let $p := n - 2$ and $y := x_{n-1}$. We have $y x_1$, $x_p y \in D$ and $x_p x$, $x x_1 \in D$ by Claim 1(iii). Therefore, y is a T-vertex and $d(y) = n - 1$.

Claim 2. At least two vertices of C are not adjacent to x unless D is isomorphic to D_5 or D_7 or belongs to the set $L_1 \cup L_2$.

Proof. We prove Claim 2 by contradiction. Let $C := x_1 x_2 \ldots x_{n-1} x_1$. Then, by Lemma 1, $d(x) = n - 1$ and $d^+(x) = d^-(x) = m$, since D is not hamiltonian. It is easy to see that some vertex x_i (say, $y := x_{n-1}$) is not adjacent to x. Then, by Claim 1(iii), $x_p x$, $x x_1 \in D$. If y is not a T-vertex, then the cycle $x_1 x_2 \ldots x_{n-2} y x_1$ contains all T-vertices. So, we can assume that y is a T-vertex. Then $d(y) = n - 1$ (by Lemma 1) and $d^+(y) = d^-(y) = m$. From our assumption it follows that

$$N^+(x) = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad N^-(x) = \{x_m, x_{m+1}, \ldots, x_p\}. \tag{1}$$

We first prove that there is a vertex x_k, $k \in [2, p-1]$, which is not adjacent to y. Assume that it is not the case. Then

$$N^+(y) = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad N^-(y) = \{x_m, x_{m+1}, \ldots, x_p\}. \tag{2}$$

Since D is not hamiltonian we have

$$A(\{x_1, \ldots, x_{m-1}\} \rightarrow \{x_{m+1}, \ldots, x_p\}) = \emptyset, \tag{3}$$

for otherwise, if $x_i x_j \in D$, where $i \in [1, m-1]$ and $j \in [m+1, n-2]$, then by (1) and (2), $H = x_1 \ldots x_i x_j \ldots x_p x x_{i+1} \ldots x_{j-1} y x_1$ is a hamiltonian cycle. Therefore
$$A(\{x_1, \ldots, x_{m-1}, x, y\} \rightarrow \{x_{m+1}, \ldots, x_{n-2}\}) = \emptyset,$$
i.e., D belongs to the set L_2 which is a contradiction.

Thus, there is a vertex x_k with $k \in [2, p-1]$ which is not adjacent to y. By Claim 1(iii), $x_{k-1} y$, $y x_{k+1} \in D$. Observe that x_k also is a T-vertex. If $k \in [m+1, p-1]$, then $m \geq 3$ and from $d^-(x_k, \{x, y\}) = 0$ it follows that there is a vertex x_i, $i \in [1, m-1]$, such that $x_i x_k \in D$. Therefore $H = x_1 \ldots x_i x_k \ldots x_p x x_{i+1} \ldots x_{k-1} y x_1$, a contradiction. So, we can assume that $k \leq m$. Similarly, we can assume that $k \geq m$. Therefore it remains to consider the case when $m = k$ and the vertex y is adjacent to all vertices of $P \setminus \{x_m\}$. If $n = 5$, i.e., $m = 2$, then $x_1 y$, $y x_3 \in D$ and $x_2 x_1 \notin D$, $x_3 x_2 \notin D$, i.e., D is isomorphic to the well-known digraph

D_5 or $D \in L$, since if we add the arc x_1x_3 or x_3x_1 (or both) to D_5, then the resulting digraph also is not hamiltonian, i.e., $D \in L_1$. Assume that $m \geq 3$. It is not difficult to see that

$$d(x_m, \{x_1, x_p\}) = 0 \text{ and } A(\{x_1, \ldots, x_{m-2}\} \to x_m) = A(x_m \to \{x_{m+2}, \ldots, x_p\}) = \emptyset, \quad (4)$$

in particular, x_m is not adjacent to x_1 and x_p. Therefore

$$\{x_{m+1}, \ldots, x_{p-1}\} \to x_m \to \{x_2, \ldots, x_{m-1}\}. \quad (5)$$

This implies that x_p and x_1 are T-vertices, since $x_1 \ldots x_{m-1}yx_{m+1} \ldots x_{p-1}x_mxx_1$ (respectively, $x_2 \ldots x_{m-1}y\ x_{m+1} \ldots x_pxx_mx_2$) is a cycle of length $n-1$ which does not contain x_p (respectively, x_1).

Now we consider the vertex y. If $x_{p-1}y \in D$, then $xx_p \notin D$ and $yx_p \notin D$ imply that $x_ix_p \in D$ for some $i \in [1, m-1]$, and hence $H = x_1 \ldots x_ix_pxx_{i+1} \ldots x_{p-1}yx_1$, a contradiction. So, we can assume that $x_{p-1}y \notin D$ and, similarly, $yx_2 \notin D$, i.e., $yx_{p-1}, x_2y \in D$. Using Lemma 2, we obtain that

$$\{x_1, x_2, \ldots, x_{m-1}\} \to y \to \{x_{m+1}, x_{m+2}, \ldots, x_p\}. \quad (6)$$

It is not difficult to see that $d^+(x_1, P[x_3, x_{m+1}]) = 0$, for otherwise, if $x_1x_i \in D$, $i \in [3, m]$, then by (1) and (6), $H = x_1x_i \ldots x_pxx_2 \ldots x_{i-1}yx_1$, and if $x_1x_{m+1} \in D$, then by (1), (5) and (6), $H = x_1x_{m+1} \ldots x_pxx_mx_2 \ldots x_{m-1}yx_1$, which is a contradiction. Similarly, we can show that $d^-(x_p, P[x_{m-1}, x_{p-2}]) = 0$. Therefore

$$N^+(x_1) = \{x_2, y, x_{m+2}, x_{m+3}, \ldots, x_p\} \text{ and } N^-(x_p) = \{x_{p-1}, y, x_1, x_2, \ldots, x_{m-2}\}. \quad (7)$$

By (7), (5) and (6) it is easy to see that $x_1 \ldots x_{m-2}x_pyx_{m+1} \ldots x_{p-1}x_mxx_1$ (respectively, $x_1x_{m+2} \ldots x_pxx_m\ x_2 \ldots x_{m-1}yx_1$) is a cycle of length $n-1$, which does not contain x_{m-1} (respectively, x_{m+1}). This means that x_{m-1} and x_{m+1} are T-vertices.

Now we will consider the vertex x_{m-1}. Then $x_{m-1}x_i \notin D$ for all $i \in [m+2, p]$ (for otherwise, by (5), $H = x_1 \ldots x_{m-1}x_i \ldots x_pyx_{m+1} \ldots x_{i-1}x_mxx_1$) and $x_{m-1}x_1 \notin D$ (for otherwise, $H = x_1 \ldots x_{m-2}yx_{m+1} \ldots x_pxx_mx_{m-1}x_1$ by (5) and (6)). Thus, we have $d^+(x_{m-1}, \{x_1, x, x_{m+2}, \ldots, x_p\}) = 0$. Therefore

$$x_{m-1} \to \{x_2, \ldots, x_{m-2}, y, x_m, x_{m+1}\}. \quad (8)$$

Now, if $m \geq 4$, then by (7), (1), (8) and (5) we have $H = x_1x_pxx_{m-1}x_{m+1} \ldots x_{p-1}x_mx_2 \ldots x_{m-2}yx_1$, which is a contradiction.

Therefore $m = 3$, i.e., $n = 7$. From (4), (5) and (7) we obtain that $x_4x_3, x_3x_2, x_1x_5 \in D$, x_1 and x_5 are T-vertices and $d(x_3, \{x_1, x_5\}) = 0$. It is easy to see that $d^+(x_2, \{x_1, x_5\}) = d^+(x_5, \{x_2, x_4\}) = 0$. From this we conclude that $x_5x_1 \in D$. Now we see that $x_1x_5yx_4x_3xx_1$ is a cycle of length $n-1$ which does not contain x_2. This means that x_2 is a T-vertex and $d^+(x_2) = d^-(x_2) = 3$. Since $d^+(x_2, \{x, x_1, x_5\}) = 0$, it follows that $x_2x_4 \in D$. Therefore D is isomorphic to the digraph D_7. Claim 2 is proved. $\square$

Claim 3. Let $x_{p-1}x, yx_p \in D$ and for some $k \in [2, p-2]$ x_k and y are not adjacent. Then x_k and x_p also are not adjacent.

Proof. Since x_k and y are not adjacent it follows that $x_{k-1}y, yx_{k+1} \in D$ (by Claim 1(iii)). Now if $x_kx_p \in D$, then $H = x_1 \ldots x_kx_pyx_{k+1} \ldots x_{p-1}xx_1$; and if $x_px_k \in D$, then $H = x_1 \ldots x_{k-1}yx_px_k \ldots x_{p-1}xx_1$. In each case we have obtained a hamiltonian cycle, which is a contradiction. $\square$

Claim 4. If $x_{p-1}x$ and $yx_p \in D$, then $d(x_i, \{x, y\}) \geq 1$ for all $i \in [2, p-2]$.

Proof. Suppose, on the contrary, that $d(x_i, \{x, y\}) = 0$ for some $i \in [2, p-2]$. Then by Claim 1(iii), $x_{i-1} \to \{x, y\} \to x_{i+1}$, and by Claim 3 the vertices x_i and x_p are not adjacent. Now, since x_i is a T-vertex and cannot be inserted into $P[x_1, x_{i-1}]$ and into $P[x_{i+1}, x_{p-1}]$, using Lemma 2, we obtain that

$$p + 1 = d(x_i) = d(x_i, P[x_1, x_{i-1}]) + d(x_i, P[x_{i+1}, x_{p-1}]) \leq i + p - i = p,$$

a contradiction. $\square$

Claim 5. If $x_{p-1}x \in D$, then the vertices y and x_{p-1} are adjacent.

Proof. Suppose, on the contrary, that y and x_{p-1} are not adjacent. Then by Claim 1(iii), $x_{p-2}y, yx_p \in D$. If $x_px_{p-1} \in D$, then $H = x_1 \ldots x_{p-2}yx_px_{p-1}xx_1$, a contradiction. So, we can assume that $x_px_{p-1} \notin D$. Moreover, if $xx_i \in D$ with $i \in [2, p-2]$, then $x_{i-1}x_{p-1} \notin D$ (for otherwise, we would have a hamiltonian cycle $H = x_1 \ldots x_{i-1}x_{p-1}x_pxx_i \ldots x_{p-2}yx_1$). Recall (by Claim 2) that there is a vertex x_l with $l \in [2, p-2]$ which is not adjacent to x. Note that $x_{l-1}x$ and $xx_{l+1} \in D$ by Claim 1(iii). Since x is a T-vertex, it follows that $d^+(x, P[x_2, x_{p-2}]) \geq m - 2$. If we consider the vertex x_{p-1}, then from $d^-(x_{p-1}, \{y, x_p\}) = 0$ and the above observation it follows that

$$xx_{p-1} \quad \text{and} \quad x_{l-1}x_{p-1} \in D. \tag{9}$$

Hence $x_px_l \notin D$ (for otherwise, if $x_px_l \in D$, then $H = x_1 \ldots x_{l-1}xx_{p-1}x_px_l \ldots x_{p-2}yx_1$).

Case 5.1. $l \leq p - 3$. Then it is not difficult to see that the vertices x_l and x_{p-1} are not adjacent. Indeed, if $x_{p-1}x_l \in D$, then $H = x_1 \ldots x_{l-1}x_{p-1}x_l \ldots x_{p-2}yx_pxx_1$ by (9); and if $x_lx_{p-1} \in D$, then $H = x_1 \ldots x_lx_{p-1}x_pxx_{l+1} \ldots x_{p-2}yx_1$, which is a contradiction. From this,

$$p + 1 = d(x_l) = d(x_l, P[x_1, x_{l-1}]) + d(x_l, P[x_{l+1}, x_{p-2}]) + d(x_l, \{y, x_p\}). \tag{10}$$

Now we show that

$$x_lx_p \text{ and } x_{p-2}x_l \in D. \tag{11}$$

Let first $yx_l \in D$. Then $x_lx_1 \notin D$ (for otherwise, $H = x_1 \ldots x_{l-1}xx_{l+1} \ldots x_pyx_lx_1$ is a hamiltonian cycle, a contradiction). Since the vertex x_l cannot be inserted into $P[x_1, x_{l-1}]$ and $P[x_{l+1}, x_{p-2}]$, from (10), $x_px_l \notin D$ and Lemma 2 it follows that $d(x_l, P[x_1, x_{l-1}]) = l - 1$, $d(x_l, P[x_{l+1}, x_{p-2}]) = p - l - 1$ and $x_lx_p, x_{p-2}x_l \in D$.

Let next $yx_l \notin D$. Similarly as in the case $yx_l \in D$ we deduce that $d(x_l, P[x_{l+1}, x_{p-2}]) = p - l - 1$ and $x_lx_p, x_{p-2}x_l \in D$. (11) is proved.

Now using (9) and (11), we obtain a hamiltonian cycle $H = x_1 \ldots x_{l-1}x_{p-1}xx_{l+1} \ldots x_{p-2}x_lx_pyx_1$, which is a contradiction.

Case 5.2. $l = p - 2$. Then $x_px_{p-2} \notin D$ and $d(x_{p-2}, \{x_{p-1}, x_p\}) \leq 2$. By the considered case $l \leq p - 3$, w.l.o.g. we can assume that the vertex x is adjacent to all vertices of $P[x_1, x_{p-3}]$. Then

$$N^+(x) = \{x_1, x_2, \ldots, x_{m-1}, x_{p-1}\} \text{ and } N^-(x) = \{x_{m-1}, x_m, \ldots, x_{p-3}, x_{p-1}, x_p\}. \tag{12}$$

This together with $\{x_{p-3}, x_{p-1}, x_p\} \to x$ implies that $m \geq 3$ and $xx_2 \in D$.

Subcase 5.2.1. $yx_2 \in D$. Assume that $yx_{p-2} \notin D$. Then $d^+(y, P[x_2, x_{p-3}]) = m - 2$ since y and x_{p-1} are not adjacent. From this and $d^-(x_{p-2}, \{x, y, x_p\}) = 0$ it follows that $x_ix_{p-2}, yx_{i+1} \in D$ for some $i \in [1, p-4]$. Therefore $H = x_1 \ldots x_ix_{p-2}x_{p-1}x_pyx_{i+1} \ldots x_{p-3}xx_1$, which is a contradiction. So, we can assume that $yx_{p-2} \in D$. Now it is easy to see that x_1

and x_{p-2} are not adjacent. Indeed, if $x_1 x_{p-2} \in D$, then $H = x_1 x_{p-2} x_{p-1} x_p y x_2 \ldots x_{p-3} x x_1$; and if $x_{p-2} x_1 \in D$, then $H = x_1 \ldots x_{p-3} x x_{p-1} x_p y x_{p-2} x_1$; which is a contradiction. Since x_{p-2} cannot be inserted into $P[x_2, x_{p-3}]$, by Lemma 2 we have $d(x_{p-2}, P[x_2, x_{p-3}]) \leq p - 3$. On the other hand,

$$p + 1 = d(x_{p-2}) = d(x_{p-2}, P[x_2, x_{p-3}]) + d(x_{p-2}, \{x_{p-1}, x_p\}) + a(x_{p-2}, y)$$

implies that $d(x_{p-2}, P[x_2, x_{p-3}]) = p - 3$. Hence, by Lemma 2, $x_{p-2} x_2 \in D$ and $x_2 \ldots x_{p-3} x x_{p-1} x_p y x_{p-2}\, x_2$ is a cycle of length $n - 1$ which does not contain x_1. Therefore x_1 is a T-vertex. Now we consider the vertex x_1. Observe that if $x_1 x_i \in D$, $i \in [m, p-2]$, then by (12), $H = x_1 x_i \ldots x_p y x_2 \ldots x_{i-1} x x_1$; and if $x_1 x_{p-1} \in D$, then $H = x_1 x_{p-1} x_p y x_{p-2} x_2 \ldots x_{p-3} x x_1$, a contradiction. Therefore $d^+(x_1, \{x, y, x_m, x_{m+1}, \ldots, x_{p-1}\}) = 0$ which contradicts that x is a T-vertex.

Subcase 5.2.2. The vertices x_2 and y are not adjacent. Then $x_1 y, y x_3 \in D$ by Claim 1(iii), and by Claim 3 the vertices x_2 and x_p also are not adjacent. Observe that if $x_i x \in D$ with $i \in [3, p-1]$, then $x_2 x_{i+1} \notin D$ (for otherwise, $H = x_1 x_2 x_{i+1} \ldots x_p y x_3 \ldots x_i x x_1$). From this we have, if $x_2 x \notin D$, then $d^-(x, P[x_3, x_{p-1}]) = m - 1$ and at least $m + 2$ vertices are not dominated by x_2 since $d^+(x_2, \{y, x, x_1\}) = 0$, which contradicts that x_2 is a T-vertex. So, we can assume that $x_2 x \in D$. Since the vertex x is adjacent to all vertices of $P[x_1, x_{p-3}]$ it follows that $m = 3$. Note that $x_2 x_4 \in D$ by (9), and x_2, x_3, x_4 are T-vertices. It is easy to see that

$$d^+(x_2, \{x_1, x_5, y\}) = d^+(x_3, \{x, x_1, x_2\}) = d^+(x_4, \{y, x_3, x_1\}) = d^-(x_3, \{x, x_4, x_5\}) = 0.$$

Therefore $x_3 x_5, x_4 x_2, x_1 x_3 \in D$. Since $x_1 y x_3 x_4 x_2 x x_1$ (respectively, $x_2 x_3 y x_5 x x_4 x_2$) is a cycle of length $n - 1 = 6$, it follows that x_5 (respectively, x_1) is a T-vertex. Now from

$$d^+(x_5, \{x_2, x_3, x_4\}) = d^-(x_1, \{x_2, x_3, x_4\}) = 0$$

we have $x_5 x_1 \in D$. Therefore, D is isomorphic to the well-known digraph D_7 or is hamiltonian, a contradiction to our assumption.

Subcase 5.2.3. $x_2 y \in D$ and $y x_2 \notin D$. Then by Claim 1(ii) we have $x_1 y \in D$ and there is a vertex x_k to $k \in [3, p-3]$ which is not adjacent to y (since $m \geq 3$). Then $x_{k-1} y$ and $y x_{k+1} \in D$ by Claim 1(iii). Using Claim 3, we obtain that x_k is not adjacent to x_1 and x_p. Since x_k cannot be inserted into $P[x_2, x_{k-1}]$ and $P[x_{k+1}, x_{p-1}]$, applying Lemma 2 to these paths, we obtain that

$$d(x_k, P[x_2, x_{k-1}]) \leq k - 1, \quad d(x_k, P[x_{k+1}, x_{p-1}]) \leq p - k,$$

$$p + 1 \leq d(x_k) = d(x_k, P[x_2, x_{k-1}]) + d(x_k, P[x_{k+1}, x_{p-1}]) + a(x_k, x)$$

and $a(x_k, x) = 2$ (in other words $x x_k, x_k x \in D$) and each inequality is, in fact, an equality. Hence, by Lemma 2, $x_k x_2, x_{p-1} x_k \in D$. From $x x_k, x_k x \in D$ we obtain that

$$N^+(x) = \{x_1, x_2, \ldots, x_k, x_{p-1}\} \text{ and } N^-(x) = \{x_k, x_{k+1}, \ldots, x_{p-3}, x_{p-1}, x_p\}$$

and $x_1 \ldots x_{k-1} y x_{k+1} \ldots x_{p-1} x_k x x_1$ is a cycle of length $n - 1$. Therefore x_p is a T-vertex and $k = m - 1$. Now we will consider the vertex x_p. Then $x_p x_i \notin D$ for all $i \in [k, p-1] \cup \{2\}$ (for otherwise, $H = x_1 x_2 \ldots x_{i-1} x x_{p-1} x_p x_i \ldots x_{p-2} y x_1$ when $i \in [k+1, p-2]$; and $H = x_1 \ldots x_{i-1} y x_p x_i \ldots x_{p-1} x x_1$ when $i = 2, k, p-1$ which is a contradiction). Thus, we have that the vertex x_p does not dominate at least $m + 1$ vertices, which is a contradiction, since x_p is a T-vertex. This contradiction completes the proof of Claim 5. $\square$

By Claim 2 there is a vertex x_l, where $l \in [2, p-1]$, which is not adjacent to x, and by Claim 1(iii), $x_{l-1} x, x x_{l+1} \in D$.

Remark 1. Let a vertex x_k, where $k \in [2, p-1]$, is not adjacent to the vertices x and y (in other words $d(x_k, \{x, y\}) = 0$). Then $x_p x_k, x_k x_1 \in D$ and $N^-(x) = N^-(y)$, $N^+(x) = N^+(y)$.

By Claim 1(iii), $x_{k-1} \rightarrow \{x, y\} \rightarrow x_{k+1}$, x_k is a T-vertex and x_k cannot be inserted into $P[x_1, x_{k-1}]$ and $P[x_{k+1}, x_p]$. Using Lemma 2, we obtain that

$$d(x_k, P[x_1, x_{k-1}]) \leq k \quad \text{and} \quad d(x_k, P[x_{k+1}, x_p]) \leq p - k + 1,$$

$$p + 1 = d(x_k) = d(x_k, P[x_1, x_{k-1}]) + d(x_k, P[x_{k+1}, x_p]) \leq p + 1.$$

Therefore, each inequality is, in fact, an equality. Hence, by Lemma 2, $x_p x_k, x_k x_1 \in D$.

Now we show that $N^-(x) = N^-(y)$ and $N^+(x) = N^+(y)$. Assume that this is not the case. Let $x_i x \in D$ and $x_i y \notin D$. Then $x_i \notin \{x_{k-1}, x_p\}$, and by Claim 1(ii), $y x_{i+1} \in D$. Since $x_k x_1, x_k x_p \in D$, it is not difficult to see that $H = x_1 x_2 \ldots x_i x x_{k+1} \ldots x_p y x_{i+1} \ldots x_k x_1$ when $i < k - 1$ and $H = x_1 x_2 \ldots x_{k-1} y x_{i+1} \ldots x_p x_k \ldots x_i x x_1$ when $i > k$, a contradiction. To show that $N^+(x) = N^+(y)$ it suffices to consider the converse digraph of D. $\square$

Claim 6. $d^+(x_{p-1}, \{x, y\}) \leq 1$.

Proof. Suppose, on the contrary, that $x_{p-1} x$ and $x_{p-1} y \in D$. Then $l \leq p - 2$. Since D is not hamiltonian it follows that if $x x_{i+1} \in D$ or $y x_{i+1} \in D$, then $x_i x_p \notin D$. This together with $d^-(x_p, \{x, y\}) = 0$ and $d^+(x, P[x_2, x_{p-1}]) = m - 1$, implies that at least $m + 1$ vertices do not dominate x_p. Clearly, x_p is not T-vertex. We will distinguish three cases according as $x_l y \in D$ or $x_l y \notin D$ and $y x_l \in D$ or x_l and y are not adjacent.

Case 6.1. $x_l y \in D$. Then $d^-(x_l, \{x_p, x_{p-1}\}) = 0$ (for otherwise, if $x_p x_l \in D$, then $H = x_1 \ldots x_{l-1} x x_{l+1} \ldots x_p x_l y x_1$; and if $x_{p-1} x_l \in D$, then $x_1 \ldots x_{l-1} x x_{l+1} \ldots x_{p-1} x_l y x_1$ is a T-cycle, a contradiction). So, by the above observation we have that x_p and x_l are not adjacent. Since $x_{p-1} x_l \notin D$ and the vertices x_l cannot be inserted into $P[x_1, x_{l-1}]$ and $P[x_{l+1}, x_{p-1}]$, using Lemma 2, we obtain that
$$d(x_l, P[x_1, x_{l-1}]) \leq l \quad \text{and} \quad d(x_l, P[x_{l+1}, x_{p-1}]) \leq p - l - 1.$$
Therefore
$$p + 1 = d(x_l) = d(x_l, P[x_1, x_{l-1}]) + d(x_l, P[x_{l+1}, x_{p-1}]) + a(x_l, y).$$
From this we conclude that $y x_l \in D$ and each inequality is, in fact, an equality. Hence, by Lemma 2, $x_l x_1 \in D$ and $H = x_1 \ldots x_{l-1} x x_{l+1} \ldots x_p y x_l x_1$, which is a contradiction.

Case 6.2. $x_l y \notin D$ and $y x_l \in D$. Then $x_l x_1 \notin D$ (for otherwise, $H = x_1 \ldots x_{l-1} x x_{l+1} \ldots x_p y x_l x_1$) and from $d(y) = n - 1$ by Claim 1(ii) we have, $y x_{l+1} \in D$. Since x_l cannot be inserted into $P[x_{l+1}, x_p]$ and into $P[x_1, x_{l-1}]$, using Lemma 2, we obtain that
$$d(x_l, P[x_1, x_{l-1}]) = l - 1 \quad \text{and} \quad d(x_l, P[x_{l+1}, x_p]) = p - l + 1,$$
and $x_p x_l \in D$. By Claim 2 there is a vertex x_k, where $k \in [2, p - 2]$, which is not adjacent to y. Then $x_{k-1} y, y x_{k+1} \in D$ (by Claim 1(iii)) and x_k is a T-vertex. We can assume that $x_k x \notin D$ (for otherwise, for the vertex y we would have Case 6.1).

Assume first that $k \leq l - 1$. Then from $x_k x \notin D$ it follows that $k \leq l - 2$. Now we will consider the vertex x_k. It is easy to see that $x_k x_p \notin D$ since D is not hamiltonian. Since x_p is not T-vertex and $y x_l \in D$ it follows that if $x_p x_k \in D$, then $H = x_1 \ldots x_{k-1} y x_l \ldots x_p x_k \ldots x_{l-1} x x_1$ is a hamiltonian cycle, and if $x_{p-1} x_k \in D$, then $x_1 \ldots x_{k-1} y x_l \ldots x_{p-1} x_k \ldots x_{l-1} x x_1$ is a T-cycle. In each case we have a contradiction. Therefore the vertices x_k and x_p are not adjacent and $x_{p-1} x_k \notin D$. Consequently, since x_k cannot be inserted into $P[x_1, x_{k-1}]$ and $P[x_{k+1}, x_{p-1}]$ by Lemma 2 we obtain
$$d(x_k, P[x_1, x_{k-1}]) \leq k \quad \text{and} \quad d(x_k, P[x_{k+1}, x_{p-1}]) \leq p - k - 1.$$
Therefore
$$p + 1 = d(x_k) = d(x_k, P[x_1, x_{k-1}]) + d(x_k, P[x_{k+1}, x_{p-1}]) + a(x_k, x) \leq k + p - k - 1 + 1 = p,$$
which leads to a contradiction, since $x_k x \notin D$ ($a(x_k, x) \leq 1$).

Assume second that $k \geq l + 1$. From $x_l y \notin D$ it follows that $k \geq l + 2$. We may assume that y is adjacent to all vertices of $P[x_1, x_{l+1}]$. Then

$$\{x_1, x_2, \ldots, x_{l+1}\} \subseteq N^+(y) \text{ and } d^-(y, P[x_{l+1}, x_{p-1}]) = m - 1.$$

Now consider the vertex x_l. It is not difficult to see that if $x_i y \in D$, $i \in [l+1, p-1]$, then $x_l x_{i+1} \notin D$ (for otherwise, $H = x_1 \ldots x_l x_{i+1} \ldots x_p x x_l \ldots x_i y x_1$). Therefore, since x_l is a T-vertex and $d^+(x_l, \{x, y\}) = 0$, we obtain that x_l does not dominate at least $m + 1$ vertices, which is a contradiction and completes the proof of Case 6.2.

Let $\{x_{l_1}, x_{l_2}, \ldots x_{l_r}\}$ be a set of vertices which at the same time are not adjacent to x and y, where $2 \leq l_1 < l_2 < \cdots < l_r \leq p - 1$. Note that (by Claim 1(iii)) for all $i \in [1, r]$ we have $x_{l_i-1}x, xx_{l_i+1}, x_{l_i-1}y$ and $yx_{l_i+1} \in D$.

Remark 2. The set $\{x, y, x_{l_1}, x_{l_2}, \ldots, x_{l_r}\}$ is an independent set of vertices.

Indeed, if $x_{l_i}x_{l_j} \in D$ and $l_i < l_j$, then $H = x_1 \ldots x_{l_i}x_{l_j} \ldots x_p x x_{l_i+1} \ldots x_{l_j-1}yx_1$; and if $x_{l_i}x_{l_j} \in D$ and $l_i > l_j$, then by Remark 1, $x_p x_{l_i} \in D$ and $H = x_1 \ldots x_{l_j-1}yx_{l_i+1} \ldots x_p x_{l_i}x_{l_j} \ldots x_{l_i-1}xx_1$. In each case we arrive at a contradiction. $\square$

Case 6.3. The vertices x_l and y are not adjacent. We can assume that for all $j \in [2, p-2]$ the vertices x_j and x are not adjacent if and only if x_j and y are not adjacent. Then by Remarks 1 and 2 for all $i \in [1, r]$ we have

$$N^+(x) = N^+(y) = N^+(x_{l_i}) \text{ and } N^-(x) = N^-(y) = N^-(x_{l_i}),$$

and $\{x, y, x_{l_1}, x_{l_2}, \ldots, x_{l_r}\}$ is an independent set of vertices. Note that if $xx_{i+1} \in D$, then $x_i x_p \notin D$ (for otherwise, $H = x_1 \ldots x_i x_p x x_{i+1} \ldots x_{p-1}yx_1$). From this and $d^-(x_p, \{x, y\}) = 0$ it follows that at least $m + 1$ vertices do not dominate x_p. Therefore, x_p is not a T-vertex. Similarly, we can show that if $\{x_i, x_{i+1}\} \rightarrow x$ (respectively, $x \rightarrow \{x_j, x_{j+1}\}$), then x_{i+1} (respectively, x_j) is not a T-vertex; and if $xx_i \in D$ and $x_j x \in D$, then $x_{i-1}x_{j+1} \notin D$. The proof of Claim 6 is completed. $\square$

Claim 7. $x_{p-1}, x \notin D$.

Proof. Suppose, on the contrary, that $x_{p-1}x \in D$. Then, by Claims 5 and 6 we have $x_{p-1}y \notin D$ and $yx_{p-1} \in D$. Hence by Claim 1(ii), $yx_p \in D$. From this and Claim 2 it follows that $m \geq 3$. There are three possibilities: $xx_2 \in D$ or x and x_2 are not adjacent or $x_2 x \in D$.

Case 7.1. $xx_2 \in D$. If $yx_2 \in D$ or y and x_2 are not adjacent, then for the converse digraph of D we have that Claim 5 or Claim 6 is not true. Thus, we can assume that $x_2 y \in D$ and $yx_2 \notin D$. Then $x_1 y \in D$, by Claim 1(ii). Recall that there is a vertex x_k with $k \in [3, p-2]$ (by Claim 2) which is not adjacent to the vertex y and hence, by Claim 1(iii), $x_{k-1}y, yx_{k+1} \in D$ and x_k is a T-vertex.

Now we will prove that the vertex x_k is not adjacent to the vertices x_1 and x_p and

$$x_{p-1}x_k, \; x_k x_2, \; x_k x, \; xx_k \in D. \tag{13}$$

Suppose that this is not the case. If $x_k x_1 \in D$, then $H = x_1 yx_{k+1} \ldots x_p x x_2 \ldots x_k x_1$; if $x_1 x_k \in D$, then $H = x_1 x_k \ldots x_p x x_2 \ldots x_{k-1}yx_1$; if $x_k x_p \in D$, then $H = x_1 \ldots x_k x_p yx_{k+1} \ldots x_{p-1}xx_1$; and finally if $x_p x_k \in D$, then $H = x_1 \ldots x_{k-1}yx_p x_k \ldots x_{p-1}xx_1$. In each case we have a contradiction. Therefore x_k is not adjacent to the vertices x_1 and x_p. From this it follows that (since x_k is a T-vertex)

$$p + 1 = d(x_k) = d(x_k, P[x_2, x_{k-1}]) + d(x_k, P[x_{k+1}, x_{p-1}]) + a(x_k, x). \tag{14}$$

Since the vertex x_k cannot be inserted into $P[x_2, x_{k-1}]$ and $P[x_{k+1}, x_{p-1}]$ by Lemma 2, we have

$$d(x_k, P[x_2, x_{k-1}]) \leq k - 1 \text{ and } d(x_k, P[x_{k+1}, x_{p-1}]) \leq p - k.$$

This together with (14) implies that the above inequalities, in fact, are equalities and $a(x, x_k) = 2$ (in other words $x_k x, x x_k \in D$). Again, using Lemma 2, we obtain that $x_{p-1} x_k, x_k x_2 \in D$. (13) is proved.

From (13) and Claim 2 it follows that $m \geq 4$. By (13), the cycle $x_1 \ldots x_{k-1} y x_{k+1} \ldots x_{p-1} x_k x x_1$ (respectively, $x_2 \ldots x_{k-1} y x_{k+1} \ldots x_p x x_k x_2$) has length $n - 1$ and does not contain x_p (respectively, x_1). Therefore, x_p and x_1 are T-vertices. It is easy to see that

$$\text{if} \quad y x_i \in D \quad \text{with} \quad i \in [2, p-1], \quad \text{then} \quad x_{i-1} x_p \notin D \tag{15}$$

(otherwise, if $y x_i$ and $x_{i-1} x_p \in D$, then $x_1 \ldots x_{i-1} x_p y x_i \ldots x_{p-1} x x_1$ is a hamiltonian cycle). Note that $x_{k-1} x_p \notin D$ (otherwise if $x_{k-1} x_p \in D$, then by (13), $x_1 \ldots x_{k-1} x_p y x_{k+1} \ldots x_{p-1} x_k x x_1$ is a hamiltonian cycle, a contradiction). From (15), $d^+(y, P[x_2, x_{p-1}]) = m - 2$, $x_{k-1} x_p \notin D$ and $x x_p \notin D$ it follows that at least m vertices do not dominate x_p. Consequently, the vertex y is adjacent to all vertices of $P - \{x_k\}$. Hence

$$\{x_1, x_2, \ldots, x_{k-1}\} \to y \to \{x_{k+1}, x_{k+2}, \ldots, x_p\}, \tag{16}$$

and $k - 1 = p - k = m - 1$. From $x_{k-1} x_p \notin D$ and (15),(16) we have

$$d^-(x_p, P[x_{k-1}, x_{p-2}]) = 0 \quad \text{and} \quad \{x_1, x_2, \ldots, x_{k-2}\} \to x_p. \tag{17}$$

From this and (13) we have that $x_1 \ldots x_{k-2} x_p y x_{k+1} \ldots x_{p-1} x_k x x_1$ is a cycle of length $n - 1$ which does not contain x_{k-1}. This means that x_{k-1} is a T-vertex and x_{k-1} cannot be inserted into $P[x_1, x_{k-2}]$ and $P[x_{k+1}, x_{p-1}] x_k$.

Now we will consider the vertex x_{k-1} and claim that x_{k-1} is not adjacent to the vertices x_1 and x_p. Indeed, if $x_1 x_{k-1} \in D$, then by (13), $H = x_1 x_{k-1} \ldots x_p x x_2 \ldots x_{k-2} y x_1$; if $x_{k-1} x_1 \in D$, then by (17) and (13), $H = x_1 x_p y x_{k+1} \ldots x_{p-1} x_k x x_2 \ldots x_{k-1} x_1$; if $x_p x_{k-1} \in D$, then by (16), $H = x_1 \ldots x_{k-2} y x_p x_{k-1} \ldots x_{p-1} x x_1$; if $x_{k-1} x_p \in D$, then by (13) and (16), $H = x_1 \ldots x_{k-1} x_p y x_{k+1} \ldots x_{p-1} x_k x x_1$. In each case we have obtained a contradiction. Therefore x_{k-1} is not adjacent to the vertices x_1 and x_p. Now by Lemma 2 we have

$$p + 1 = d(x_{k-1}) = d(x_{k-1}, P[x_2, x_{k-2}]) + d(x_{k-1}, P[x_{k+1}, x_{p-1}] \cup \{x_k\}) + a(x_{k-1}, \{x, y\}) \leq$$

$$p - 1 + a(x_{k-1}, \{x, y\}).$$

It is possible only if $a(x_{k-1}, \{x, y\}) = 2$ (i.e., $x_{k-1} y$ and $x x_{k-1} \in D$ since $y x_{k-1} \notin D$ and $x_{k-1} x \notin D$). It is not difficult to see that $d^-(x_1, P[x_{k-1}, x_{p-1}]) = 0$ (otherwise if $x_i x_1 \in D$, $i \in [k, p-1]$, then $H = x_1 y x_{i+1} \ldots x_p x x_2 \ldots x_i x_1$). Hence $x_{k-2} x_1 \in D$ and by (13), $H = x_1 y x_{k+1} \ldots x_p x x_{k-1} x_k x_2 \ldots x_{k-2} x_1$, which is a contradiction. The contradiction completes the proof of Case 7.1 .

Case 7.2. The vertices x and x_2 are not adjacent. Then by Claim 1(iii), $x_1 x$ and $x x_3 \in D$. By Claim 4 we have that the vertices x_2 and y are adjacent. If we consider the converse digraph of D, then using Claim 5 we see that $x_2 y \in D$ and $y x_2 \notin D$. Therefore, by Claim 1(ii), $x_1 y \in D$ since y is a T-vertex. Now we will consider the vertex x_2. Note that x_2 also is a T-vertex. If $x_p x_2 \in D$, then $H = x_1 y x_p x_2 \ldots x_{p-1} x x_1$, a contradiction. So, we can assume that $x_p x_2 \notin D$. By Lemma 2, $d(x_2, P[x_3, x_p]) \leq p - 2$ since x_2 cannot be inserted into $P[x_3, x_p]$. From this, since x and x_2 are not adjacent, $y x_2 \notin D$ and x_2 is a T-vertex, we obtain that $x_2 x_1 \in D$. Now it is easy to see that if $y x_i \in D$ with $i \in [4, p]$, then $x_{i-1} x_2 \notin D$ (for otherwise, $H = x_1 y x_i \ldots x_p x x_3 \ldots x_{i-1} x_2 x_1$). Consequently, from $d^+(y, P[x_4, x_p]) = m - 1$

and $d^-(x_2, \{x, y\}) = 0$ it follows that at least $m + 1$ vertices do not dominate x_2, which is a contradiction. The obtained contradiction completes the proof of Case 7.2 .

Case 7.3. $x_2x \in D$. Then $x_1x \in D$ by Claim 1(ii). Then from $d^-(x, \{x_1, x_2, x_{p-1}, x_p\}) = 4$ we have $m \geq 4$. It follows that there is a $l \in [3, p-2]$ such that $x_{l-2}x, x_{l-1}x, xx_{l+1} \in D$ and x_l and x are not adjacent by Claim 2. Note that with respect to the vertices x_2 and y the following subcases are possible: $yx_2 \in D$ or $x_2y \in D$ or the vertices y and x_2 are not adjacent.

Subcase 7.3.1. $yx_2 \in D$. It is not difficult to see that the vertices x_1 and x_l are not adjacent. Indeed, if $x_1x_l \in D$, then $H = x_1x_l \ldots x_pyx_2 \ldots x_{l-1}xx_1$; and if $x_lx_1 \in D$, then $H = x_1xx_{l+1} \ldots x_pyx_2 \ldots x_lx_1$, which is a contradiction.

We first prove that

$$yx_l, \; x_lx_2, \; x_lx_{l-1}, \; x_lx_{l-2} \in D \text{ and } x_{l-2}x_l \notin D. \tag{19}$$

Proof of (19). Assume that $x_px_l \in D$. Then $x_ly \notin D$ (for otherwise, if $x_ly \in D$, then $H = x_1 \ldots x_{l-1}xx_{l+1} \ldots x_px_lyx_1$). Since x_1 and x_l are not adjacent and x_l cannot be inserted into $P[x_2, x_{l-1}]$ and $P[x_{l+1}, x_p]$, using Lemma 2, we see that
$$p + 1 = d(x_l) = d(x_l, P[x_2, x_{l-1}]) + d(x_l, P[x_{l+1}, x_p]) + a(x_l, y) \leq p + a(x_l, y).$$

It follows that $d(x_l, P[x_2, x_{l-1}]) = l - 1$ and $a(x_l, y) = 1$. Therefore $yx_l \in D$ and $x_lx_2 \in D$ by Lemma 2.

Now assume that $x_px_l \notin D$. Then, similarly, as before we obtain that $d(x_l, P[x_2, x_{l-1}]) = l - 1$, $d(x_l, P[x_{l+1}, x_p]) = p - l$ and $a(x_l, y) = 2$ (i.e., $yx_l, x_ly \in D$). By Lemma 2, we have that $x_lx_2 \in D$. Now we will consider the path $x_{l+1}x_{l+2} \ldots x_pyx_1 \ldots x_{l-2}x_{l-1}$ and the vertex x_l instead of y. Then using Claims 6 and 5 we obtain that $x_lx_{l-1}, x_lx_{l-2} \in D$ and $x_{l-2}x_l \notin D$. So, indeed, (19) is satisfied, as desired. □

W.l.o.g. we can assume that $xx_{l+2} \notin D$ and x and x_{l+2} are adjacent (because otherwise for the path $x_{l+1}x_{l+2} \ldots x_pyx_1 \ldots x_{l-1}$ we would have Case 7.1 or 7.2 which we have already dealt with). Then by Claim 1(ii) we have, $x_{l+1}x, x_{l+2}x \in D$.

Now we consider the vertex x_1. If $x_ix \in D$ with $i \in [2, p-1]$, then $x_1x_{i+1} \notin D$ (for otherwise, $H = x_1x_{i+1} \ldots x_pyx_2 \ldots x_ixx_1$). If $x_1x_{l+1} \in D$, then $H = x_1x_{l+1} \ldots x_pyx_lx_2 \ldots x_{l-1}xx_1$ by (19). Observe that $x_2 \ldots x_{l-1}xx_{l+1} \ldots x_pyx_lx_2$ is a cycle of length $n - 1$ which does not contain x_1. This means that x_1 is a T-vertex. Now from $d^-(x, P[x_2, x_{p-1}]) = m - 2$ and $d^+(x_1, \{y, x_{l+1}\}) = 0$ it follows that the vertex x is adjacent to all vertices of $P - \{x_l\}$ which is not possible since $m \geq 4$, $x_{l+1}x \in D$ and D is not hamiltonian.

Subcase 7.3.2. $x_2y \in D$. Then by Claims 2 and 1(iii) there is a vertex x_k with $k \in [3, p-2]$ such that $x_{k-1}y, yx_{k+1} \in D$ and y is not adjacent to x_k. It is easy to see that x_p and x_k are not adjacent (i.e., $a(x_k, x_p) = 0$). Indeed, if $x_kx_p \in D$, then $H = x_1 \ldots x_kx_pyx_{k+1} \ldots x_{p-1}xx_1$; and if $x_px_k \in D$, then $H = x_1 \ldots x_{k-1}yx_px_k \ldots x_{p-1}x \; x_1$, which is a contradiction. Now we prove that

$$x_{p-1}x_k \text{ and } x_kx \in D. \tag{20}$$

Proof of (20). Let $x_kx_1 \in D$. Then $xx_k \notin D$ (since, otherwise, if $xx_k \in D$, then $H = x_1 \ldots x_{k-1}yx_{k+1} \ldots x_pxx_kx_1$) and hence, since $a(x_k, x_p) = 0$ and the paths $P[x_1, x_{k-1}]$ and $P[x_{k+1}, x_{p-1}]$ cannot be extended with x_k by Lemma 2 we have $d(x_k, P[x_1, x_{k-1}]) \leq k$, $d(x_k, P[x_{k+1}, x_{p-1}]) \leq p - k$ and
$$p + 1 = d(x_k) = d(x_k, P[x_1, x_{k-1}]) + d(x_k, P[x_{k+1}, x_{p-1}]) + a(x_k, x) = p + 1.$$

Therefore $d(x_k, P[x_1, x_{k-1}]) = k$, $d(x_k, P[x_{k+1}, x_{p-1}]) = p - k$ and $a(x_k, x) = 1$ (i.e., $x_kx \in D$). Now, using Lemma 2, we obtain that $x_{p-1}x_k \in D$.

Let now $x_k x_1 \notin D$. Then $d(x_k, P[x_1, x_{k-1}]) \leq k-1$, $a(x_k, x) = 2$ (i.e., $x_k x, x x_k \in D$) and $d(x_k, P[x_{k+1}, x_{p-1}]) = p - k$. Again, using Lemma 2, we obtain that $x_{p-1} x_k \in D$. So, indeed (20) is satisfied, as desired. $\square$

Now we will consider the vertex x_p which is a T-vertex since $x_1 \ldots x_{k-1} y x_{k+1} \ldots x_{p-1} x_k x x_1$ is a cycle of length $n - 1$. If $x_i y \in D$ with $i \in [1, p-2]$, then $x_p x_{i+1} \notin D$ (for otherwise, $H = x_1 \ldots x_i y x_p x_{i+1} \ldots x_{p-1} x x_1$). Note that $d^-(y, P[x_1, x_{p-2}]) = m - 1$ and $x_p x_{k+1} \notin D$ (if $x_p x_{k+1} \in D$, then by (20), $H = x_1 \ldots x_{k-1} y x_p x_{k+1} \ldots x_{p-1} x_k x x_1$. It follows from the observation above that the vertex y is adjacent to all vertices of $P - \{x_k\}$. Therefore
$$N^-(y) = \{x_1, x_2, \ldots, x_{k-1}, x_p\} \text{ and } N^+(y) = \{x_1, x_{k+1}, x_{k+2}, \ldots, x_p\}.$$

Then for the path $x_{k+1} x_{k+2} \ldots x_p x x_1 x_2 \ldots x_{k-1}$ and for the vertex y by Claims 5 and 6 we have the considered Case 7.1.

Subcase 7.3.3. The vertices y and x_2 are not adjacent. Then $x_1 y, y x_3 \in D$ (by Claim 1(iii)), x_2 and x_p are not adjacent (by Claim 3) and x_2 is a T-vertex.

Assume that $x_2 x_1 \in D$. Then $x_i x_2 \notin D$ if $x x_{i+1} \in D$, $i \in [3, p-1]$ (for otherwise, $H = x_1 x x_{i+1} \ldots x_p y x_3 \ldots x_i x_2 x_1$). Now from $d^+(x, P[x_4, x_{p-1}]) = m - 1$ and $d^-(x_2, \{x, y\}) = 0$ it follows that $d^-(x_2) \leq m - 1$, which is a contradiction. So, we can assume that $x_2 x_1 \notin D$. Therefore
$$p + 1 = d(x_2) = d(x_2, P[x_3, x_{p-1}]) + d(x_2, \{x_1, x\}) \leq d(x_2, P[x_3, x_{p-1}]) + 2.$$

Hence $d(x_2, P[x_3, x_{p-1}]) = p - 1$. By Lemma 2, x_2 can be inserted into the path $P[x_3, x_{p-1}]$, a contradiction which completes the proof of Claim 7. $\square$

Let us now complete the poof of the theorem. Since D is not hamiltonian from Claim 7 and Remark 2 it follows that for any cycle $C := x_1 x_2 \ldots x_{2m} x_1$ of length $n - 1 = 2m$ if $x \notin V(C)$ then $N^+(x) = N^-(x) = \{x_1, x_3, \ldots, x_{2m-1}\}$ and $\{x_2, x_4, \ldots, x_{2m}, x\}$ is an independent set of vertices. Therefore $K^*_{m,m+1} \subseteq D \subseteq [K_m + \overline{K}_{m+1}]^*$. The proof of the Theorem is complete. $\square$

Remark 3. Let D be a digraph with the vertex set $V(D) = \{x_1, x_2, x_3, x_4, x_5, x, y\}$ such that $N^+(x_1) = \{x_2, x_4\}$, $N^+(x_2) = \{x, y, x_3, x_5\}$, $N^+(x_3) = N^+(x) = N^+(y) = \{x_1, x_2, x_4, \}$, $N^+(x_4) = \{x, y, x_5\}$ and $N^+(x_5) = \{x, y, x_3\}$. It is easy to check that the vertices x, y, x_2, x_3 and x_4 are T-vertices and the vertices x_1 and x_5 are not T-vertices. Moreover, the digraph D is 2-strong and contains no cycle through x, y, x_2, x_3 and x_4.

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Մեծ կիսաաստիճաններ ունեցող գագաթներով անցնող ցիկլերի մասին

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Ամփոփում

Ներկա աշխատանքում ցույց է տրված, որ եթե $2m + 1$ գագաթանի կողմնորոշված գրաֆն ունի $2m$ երկարությամբ ցիկլ, ապա այդ գրաֆը, բացառությամբ մի քանի որոշակի գրաֆների, նույնպես պարունակում է կողմնորոշված ցիկլ, որն անցնում է բոլոր այն գագաթներով, որոնց կիսաաստիճանները փոքր չեն m-ից:

О циклах проходящих через вершины с большими полустепенями

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Аннотация

В настоящей работе доказано: Если $(2m + 1)$-вершинный орграф D содержит орцикл длины $2m$, то D также (кроме некоторых орграфов) содержит орцикл проходящий через все вершины с полустепенями неменьшими m.