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A Polynomial Algorithm for the Minimum Bandwidth of Interval Graphs

David H. Muradian

Institute for Informatics and Automation Problems of NAS RA
e-mail: david.h.muradian@gmail.com

Abstract

Let G be a connected graph with vertex set X and edge set U. A layout of G is a one-to-one map φ from X onto {1, 2, . . . , |X|}. The bandwidth of φ is $B_\varphi(G)$ = max $|\varphi(u) - \varphi(v)|$, where (u, v) ranges over all edges of G. The bandwidth of G, denoted by B(G), is defined as B(G) = min$B_\varphi(G)$ where φ ranges over all layouts of G. Interval graphs are the intersection graphs of a family of intervals over the real line. In this paper we show that the Bandwidth Minimization problem for interval graphs can be solved in time $O(n\Delta^2\log(\Delta))$, where n is the vertex number and Δ is the maximal degree of vertex of the interval graph.

Keywords: Graph layout, Bandwidth, Interval Graphs.

1. Introduction

The bandwidth minimization problem for graphs was first stated in 1966 by Harper ([1]), where the problem was solved for hypercubes.

Finding the bandwidth of an arbitrary graph is an NP-complete problem ([2]) and it remains NP-complete for many simple structures, e.g., for cyclic caterpillars with hair length 1, graphs in which the removal of all pendant vertices results in a simple cycle ([3]).

There are only few classes of graphs for which an efficient solution (i.e., a polynomial algorithm or analytic result) to the bandwidth problem is known. Classes of graphs the bandwidth of which can be computed efficiently are butterflies ([4]), chain graphs ([5]), caterpillars with hair length at most 2 ([6]). Another nontrivial class, for which the problem was solved efficiently, is the class of interval graphs, graphs which are the intersection graphs of a family of intervals over the real line.

The first polynomial algorithm for interval graphs was given in 1986 by the author ([7]). It was published in the Reports of NAS RA, where the algorithm is described in detail, and besides a brief proof of its correctness is given. Since this result was obtained independently and published in the following years ([8], [9], [10]), we consider it reasonable to publish the full proof of our algorithm's correctness.

2. Preliminaries

Let G be a connected graph with vertex set X and edge set U. A numbering of G is a one-to-one map φ from X onto {1, 2, . . . , $|X|$}. The bandwidth of φ is $B_\varphi(G)$= max $|\varphi(u) - \varphi(v)|$, where (u, v) ranges over all edges of G. The bandwidth of G, denoted by $B(G)$, is defined as $B(G)$ = min $B_\varphi(G)$, where φ ranges over all numberings of G. Length of an edge (u, v) is defined as $|\varphi(u) - \varphi(v)|$.

Given a graph $G(X, U)$ and its some layout φ. Let's define a new layout $\varphi_{A,B}$ – swap of two disjoint subsets A and B of X as follows. If A and B are disjoint subsets of vertices of G, at that for any $x \in A$ and $y \in B$ we have $\varphi(x) < \varphi(y)$ and $\max_{z \in B} \varphi(z) - \min_{z \in A} \varphi(z) = |A| + |B| - 1$, then

$$\varphi_{A,B}(z) = \begin{cases} \varphi(z) / \, z \in X \setminus (A \cup B) \\ \varphi(z) + |B| / z \in A \\ \varphi(z) - |A| / z \in B \end{cases}$$

Let's denote $X_\varphi[k, l] = \{x \in X / k \le \varphi(x) \le l\}$ for 1 ≤ k ≤ l ≤ n.

Let's consider an interval graph $G = (X, U)$. Let's denote by $\hat{x}$ an interval corresponding to the vertex $x \in X$. We say that an interval $\hat{x} = (a, b)$ is entirely on the left side (right side) of an interval $\hat{y} = (c, d)$ if $b < c$ (correspondingly $a.d$. Let's denote by $\Gamma^-(\hat{x})$ ($\Gamma^+(\hat{x})$) the set of intervals which entirely are on the left side (correspondingly - on the right side) of $\hat{x}$.

Let's define a layout φ_0 for the graph G. For any vertices $x, y \in X$ if $\Gamma^-(\hat{x}) = \Gamma^-(\hat{y})$ and $\Gamma^+(\hat{x}) = \Gamma^+(\hat{y})$, then $\varphi_0(x) < \varphi_0(y)$ or $\varphi_0(x) > \varphi_0(y)$. Otherwise, $\varphi_0(x) < \varphi_0(y)$ if and only if $\Gamma^-(\hat{x}) \subset \Gamma^-(\hat{y})$ or $\Gamma^-(\hat{x}) = \Gamma^-(\hat{y})$ but $\Gamma^+(\hat{x}) \subset \Gamma^+(\hat{y})$.

It is easy to check that φ_0 is well-defined by the above conditions and has the following properties.

1. If $\hat{x}$ is entirely on the left side of $\hat{y}$, then $\varphi_0(x) < \varphi_0(y)$. It is obvious by the definition.
2. If 1 ≤ i < j ≤ n and the vertices $x = \varphi_0^{-1}(i)$, $y = \varphi_0^{-1}(j)$ are adjacent, then x is adjacent to any vertex $z = \varphi_0^{-1}(k)$, where i < k ≤ j. Really, let's assume that x, z are not adjacent. Then $\hat{x}$ is entirely on the left side of $\hat{z}$. The vertices z, y should be adjacent, otherwise as x, y are adjacent, then $\hat{y}$ should be entirely on the left side of $\hat{z}$ and therefore should have a smaller number in the layout φ_0. If z, y are adjacent, then again $\hat{y}$ should have a smaller number than $\hat{z}$ because $\Gamma^-(\hat{y}) \subset \Gamma^-(\hat{z})$ (at least on account of $\hat{x}$), which leads to a contradiction.
3. Let x and y be vertices for which we have $\Gamma^-(\hat{y}) \subset \Gamma^-(\hat{x})$ and $\Gamma^+(\hat{y}) \subset \Gamma^+(\hat{x})$. Then two disjoint intervals exist, one of which is entirely on the left side and the other – entirely on the right side of $\hat{x}$ and both are overlaps with $\hat{y}$. It is obvious by definition.

Let $\hat{x}$, $\hat{y}$, $\hat{z_1}$ and $\hat{z_2}$ be intervals where $\hat{z_1}$ is entirely on the left side of $\hat{x}$ and $\hat{z_2}$ - entirely on the right side of $\hat{x}$ and all of them overlap with $\hat{y}$. Then we will say that $\hat{x}$ is a proper interval for $\hat{y}$ and record this fact as $x \dot{\subset} y$.

For any vertex $x \in X$, we will name $\varphi_0(x)$ as its index.

3. Algorithm

Step i ($i \geq 1$). At step i the algorithm having as an input the layout $\varphi = \varphi_{i-1}$, creates a layout φ_i or stops, claiming that $B(G) > K$.

Let y be a vertex with the greatest number at φ, which is incident to an edge with a length above K, and let (x, y) have the greatest length among them (i.e., y is not adjacent to vertices, the numbers of which are less than $\varphi(x)$). Then the algorithm tries to find a vertex with the smallest number from $X_\varphi[\varphi(x) + 1, \varphi(y) - 1]$, which is not adjacent to y. If such vertex does not exist, then it claims that $B(G) > K$.

Let z be the sought-for vertex. Let's denote $M = X_\varphi[\varphi(x), \varphi(z) - 1]$. Let a_j be a vertex from the set $M \backslash X_\varphi\left[\varphi\left(a_{j-1}\right), \varphi(z) - 1\right]$ having the greatest index there (with an agreement, that $X_\varphi[\varphi(a_0), \varphi(z) - 1] = \emptyset$). Denote $S_j = X_\varphi\left[\varphi\left(a_j\right) + 1, \varphi\left(a_{j-1}\right) - 1\right]$. Note that for some j-s sets S_j may be empty. The layout φ_i is defined as follows:

$$\varphi_i(a) = \begin{cases} \varphi_{i-1}(x) / \, a = z \\ \varphi_{i-1}(a) / a \in \cup_j S_j \cup \left(X \backslash (M \cup \{z\})\right) \\ \varphi_{i-1}(a) + 1 + \left|S_j\right| / a = a_j \end{cases}$$

In other words φ_i is obtained from φ_{i-1} via successive swaps: (M,$\{z\}$), $(\{a_1\}, S_1)$, $(\{a_2\}, S_2)$, … Then the algorithm checks if there is an edge with the length above K at φ_i. If not, then it stops, claiming that φ_i is the sought-for layout with the bandwidth no more than K. If yes, then the algorithm turns to the step $i + 1$.

4. Proof of the Algorithm's Correctness

Let's define a class of layouts Φ_0 for the graph G: $\varphi \in \Phi_0$ if and only if for any vertices a and b, for which $\varphi(a) < \varphi(b)$ and $\varphi_0(a) > \varphi_0(b)$, will take place $a \dot{\subset} b$.

From the definition of Φ_0 it follows immediately that for any $\varphi \in \Phi_0$ and two non- adjacent vertices x and y, if $\varphi_0(x) < \varphi_0(y)$, then $\varphi(x) < \varphi(y)$. It is easy to see that $\varphi_0 \in \Phi_0$. Moreover, the algorithm, beginning with φ_0, never leaves the class Φ_0.

Lemma 1: *$\varphi_i \in \Phi_0$ in each step i.*

Proof: We will prove the statement by induction. For $i = 0$ the statement obviously is true, let it be true for each step until $i - 1$. Let's show that $\varphi_i \in \Phi_0$. Then it is sufficient to show that $z \dot{\subset} q$ any $q \in M$, and $a_t \dot{\subset} q_t$ for all $q_t \in S_t$.

Since $(q, y) \in U$ and $(z, y) \notin U$, then naturally q cannot be a proper interval for z, and by the fact that $\varphi_{i-1} \in \Phi_0$, we will have $\varphi_0(q) < \varphi_0(z)$. But as $\Gamma^+(\hat{q}) \subset \Gamma^+(\hat{z})$, it follows immediately that $\Gamma^-(\hat{q}) \subset \Gamma^-(\hat{z})$, i.e., $z \dot{\subset} q$. Then the layout obtained by the swap of M and {z} certainly belongs to Φ_0, and therefore, $a_t \dot{\subset} q_t$ for all $q_t \in S_t$. This proves the lemma.

In the next two lemmas several new properties of the layout φ_i are proved.

Lemma 2: *In the step i, the length of the edge (x, y) decreases by 1 and none of vertices from $X_{\varphi_i}[\varphi_i(y) + 1, n]$ is incident to an edge with the length above K.*

Proof: By Lemma 1 we have $z \dot{\subset} x$, and z together with x don't have adjacent vertices in $X_{\varphi_i}[\varphi_i(y) + 1, n]$. Therefore, there is no vertex from $X_{\varphi_i}[\varphi_i(y) + 1, n]$, incident to an edge with length above K. Now we will show that the length of the edge (x, y) decreases exactly by 1. If not, then denoting $= \varphi_{i-1}^{-1}(\varphi_{i-1}(x)+1)$, we will have $x \dot{\subset} u$. By definition u doesn't have adjacent vertices in $X_{\varphi_{i-1}}[\varphi_{i-1}(y)+1,n]$. Let j be the greatest number, for which $\varphi_j(u) < \varphi_j(x)$ and $\varphi_{j+1}(u) > \varphi_{j+1}(x)$. It is clear that j ≤ i-2. In the sub-step j+1 some set U swaps with {x}, where u∊U, at that there exists a vertex v, which is not adjacent to x, but is adjacent to all vertices from U, and therefore, v∊$X_{\varphi_{i-1}}[\varphi_{i-1}(y)+1,n]$. Obtained contradiction proves the lemma.

Lemma 3: *Let vertices a and b satisfy the conditions:*

$$\varphi_i(a) < \varphi_i(b) \quad \text{and} \quad \varphi_0(a) > \varphi_0(b). \qquad (1)$$

Then there are vertices D and d, such that:

$$D \epsilon X_{\varphi_i}[\varphi_i(a)+1, \varphi_i(b)] \quad \text{and} \quad \varphi_i(d) - \varphi_i(D) \geq K, \qquad (2)$$

Proof: We will prove the statement by induction. For $i = 0$ the statement obviously is true, let it is true from 1 to i-1 step inclusive. Consider the step i. Denote $M_1 = X_{\varphi_{i-1}}[1, \varphi_{i-1}(x)-1]$ and $M_2 = X_{\varphi_{i-1}}[\varphi_{i-1}(z)+1, n]$.

We will show, that if for some vertices a and b conditions (1) are fulfilled, then there should exist vertices *D*' and *d*' which satisfy the conditions (2).

The only possible case, when $\varphi_{i-1}(a) > \varphi_{i-1}(b)$ can hold when a = z and b∊M. In this case vertices x and y can be taken instead of D and d. Really, any vertex from M is adjacent to $d = y$ and $\varphi_i(d) - \varphi_i(D) = \varphi_i(y) - \varphi_i(x) \geq K$. In other cases $\varphi_{i-1}(a) < \varphi_{i-1}(b)$.

Let's analyze possible cases.

Case 1. $b \epsilon M_1$.

Then $a \epsilon M_1$. Since at the step i only the numbers of vertices from $M \cup \{z\}$ can be changed, then at d'$\epsilon M_1 \cup M_2$ taking $D = D'$ and $d = d'$, it is easy to see that for D and d at φ_i conditions (2) are fulfilled. If $d' = z$, then from the fact that $z \dot{\subset} q$ for any q∊M, it follows that each vertex from $X_{\varphi_{i-1}}[\varphi_{i-1}(D'), \varphi_{i-1}(b)]$ is adjacent to each vertex from M. Note that some vertex from M receives the number $\varphi_{i-1}(z)$ at φ_i. Then taking $D = D'$ and $d = \varphi_i^{-1}(\varphi_{i-1}(z))$, one can observe, that for D and d conditions (2) will be fulfilled. If d'∊M, then as $\varphi_i(q) \geq \varphi_{i-1}(q)$ for each q∊M, we will have $\varphi_i(d') - \varphi_i(D') \geq \varphi_{i-1}(d') - \varphi_{i-1}(D')$ and we can take $D = D'$ and $d = d'$

Case 2. $b = z$.

Then $a \epsilon M_1$. If $D' \in M_1$, then we will put $D = D'$ and $d = d'$, and if $D' \in M$, then $D = b = z$ and $d = d'$. Then from the fact that $z \dot{\subset} q$ for each $q \in M$, it follows that for D and d at φ_i conditions (2) are fulfilled.

Case 3. $b \in M \cup M_1$.

At first let's assume that for any vertex from $X_{\varphi_{i-1}}[\varphi_{i-1}(x), \varphi_{i-1}(b)-1]$ occurs $\varphi_0(q) < \varphi_0(b)$. Then $a \epsilon M_1$. If $b \in M$, then in place of D one can take the vertex x and in place of d – vertex y, i.e., all vertices from M are adjacent to y. If $b \in M_2$ and $D' \in M_2$, then during the transition from φ_{i-1} to φ_i, nothing is changed for a and b, therefore we can take $D = D'$ and $d = d'$. Let $b \in M_2$ and $D' \in M$. Since z is adjacent to d', then all vertices from M will be adjacent to d', and one can take $D = z$ (certainly only after receiving the number $\varphi_{i-1}(x)$ by z) and $D = D'$.

Now let q be a vertex from $X_{\varphi_{i-1}}[\varphi_{i-1}(x), \varphi_{i-1}(b)-1]$, the index of which is greater than the index of b, and let c be the vertex with the greatest number among them at φ_{i-1}.

Let $b \in M_2$. Then $\in M_2 \cup \{z\}$. Really, if $c \in M$, then from $z \dot{\subset} c$ we will have $\varphi_0(z) > \varphi_0(c)$ and therefore – $\varphi_0(z) > \varphi_0(b)$, which will be at odds with the selection of c. But if c∊M₂∪{z}, then at φ_{i-1} there are D' and d', satisfying (2), at that $D' \in M_2$, and as their numbers are not changed during the transition from φ_{i-1} to φ_i, then one can take $D = D'$ and $d = d'$.

Let $b \in M$. If $a \in M_1$, then one can take $D = x$ and $d = y$. Let $a \in M$. Let's analyze the step i. The inequality $\varphi_i(a) < \varphi_i(b)$ means that at φ_{i-1} there was a vertex c', such that $\varphi_{i-1}(a) < \varphi_{i-1}(c') < \varphi_{i-1}(b)$, whereas at φ_i : $\varphi_i(c') > \varphi_i(b)$, i.e., the vertex was belonging to some nonempty set S_r, and c' = a_r. Therefore, considering the pair of vertices (c', b) at φ_{i-1}, by the inductive conjecture there are vertices D' and d', satisfying (2), the numbers of which are not changed during the transition from φ_{i-1} to φ_i, i.e., one can take $D = D'$ and $d = d'$. This proves the lemma.

Before proving that the algorithm stops without creating a layout with the bandwidth K only on graphs having the bandwidth above K, we will define a graph called a generalized 1-caterpillar.

Definition: *Let A_i, V_i $(i \in \overline{1,m})$ be disjoint sets and $A_i \neq \emptyset$ for all $(i \in \overline{1,m})$. A graph H = (F,E) with the set of vertices $F = \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{m} V_i$ is called a generalized 1-caterpillar if (x,y)∈E if and only if $x \in V_i,\ y \in A_i$ $(i \in \overline{1,m})$, or $x \in A_i$, $x \in A_{i+1}$ $(i \in \overline{1,m-1})$, or $x, y \in A_i$ $(i \in \overline{1,m})$.*

Lemma 4: *If the number of vertices of the graph H is more than $(m+1)(K+1) - \sum_{i=1}^{m} A_i$, then B(H) > K.*

Proof: Let p be the number of vertices of H and $p > (m+1)(K+1) - \sum_{i=1}^{m} A_i$. Let's assume that $B(H) \leq K$. Let φ be a layout with the smallest bandwidth for H, and without losing generality, let's assume that $\varphi^{-1}(1) \in A_i \cup V_i$ and $\varphi^{-1}(p) \in A_j \cup V_j$ at some i, j $(1 \leq i \leq j \leq m)$.

Using the assumption B(H) ≤ K, it is easy to prove the following statement: if $z_1 \epsilon A_t$ for some t, and z_2 – vertex from A_{t+1} with the smallest number, then $\varphi(z_2) \leq \varphi(z_1) + K - |A_{t+1}| + 1$.

Solving this recurrent inequalities we will obtain that if z is a vertex from A_j with the smallest number, then $\varphi(z) \leq (j-i+1)(K+1) - \sum_{t=i}^{j} |A_t| + 1$. But z is adjacent to $\varphi^{-1}(p)$, therefore

$$K \geq p - \varphi(z) > (m+1)(K+1) - \sum_{t=1}^{m} |A_t| - (j-i+1)(K+1) + \sum_{t=i}^{j} |A_t| =$$
$$(m-j+i)(K+1) - \sum_{t=1}^{i-1} |A_t| - \sum_{t=j+1}^{m} |A_t| - 1.$$

Besides $|A_t| \leq K+1$ for all $t \epsilon \overline{1,m}$, therefore:

$$K \geq p - \varphi(z) > (m-j+i)(K+1) - (m-j+i-1)(K+1) - 1 = K,$$

which leads to a contradiction. This proves the lemma.

Now we will state the last necessary property of layouts from Φ_0.

Lemma 5: *Let $\varphi \epsilon \Phi_0$ and let u_1, u_2, u_3 be vertices with the following properties: $\varphi(u_1) < \varphi(u_2) < \varphi(u_3)$, $(u_1, u_3) \epsilon U$ and $\varphi_0(u_2) < \varphi_0(u_3)$. Then $(u_1, u_2) \epsilon U$.*

Proof: Let's assume that u_1, u_2 are not adjacent. Then from $\varphi(u_1) < \varphi(u_2)$ we will have $\varphi_0(u_1) < \varphi_0(u_2)$. But $(u_1, u_3) \epsilon U$ and therefore $\Gamma^-(\widehat{u_3}) \subset \Gamma^-(\widehat{u_3})$ and $\varphi_0(u_3) < \varphi_0(u_2)$, which contradicts the conditions of the lemma. This proves the lemma.

Theorem: *If the algorithm at some step stops without creating for the graph G a layout with bandwidth K, then B(G) > K.*

Proof: Let's assume that the situation described in the formulation of the theorem occurs at step i+1: each vertex from $X_{\varphi_i}[\varphi_i(x),\varphi_i(y)]$ is adjacent to y. If there is no any vertex among them, the index of which is greater than $\varphi_0(y)$, then by Lemma 5, the subgraph induced by the set $X_{\varphi_i}[\varphi_i(x),\varphi_i(y)]$ is a clique with the vertex number over K+1 and therefore B(G) > K.

Let's assume that there is a vertex in $X_{\varphi_i}[\varphi_i(x),\varphi_i(y)-1]$, the index of which is greater than $\varphi_0(y)$ and let a_1 have the greatest number among them. Denote $b_1=y$. Then for a_1 and $b_1$conditions (1) are fulfilled and therefore there exist vertices D_1 from $X_{\varphi_i}[\varphi_i(a_1)+1,\varphi_i(b_1)]$ and b_2 , such that $\varphi_i(b_2)-\varphi_i(D_1)\geq K$ and all vertices from $X_{\varphi_i}[\varphi_i(D_1),\varphi_i(b_1)]$ are adjacent to b_2. Denote $R_0=X_{\varphi_i}[\varphi_i(x),\varphi_i(D_1)-1]$ and $A_1=X_{\varphi_i}[\varphi_i(D_1),\varphi_i(b_1)]$.

Then let a_2 be the vertex with the greatest number from $X_{\varphi_i}[\varphi_i(b_1)+1,\varphi_i(b_2)]$, the index of which is equal or greater than $\varphi_0(b_2)$ (equality of indices is understood as a simple coincidence: $a_2=b_2$). Let $a_2\neq b_2$. Then a_2,b_2 satisfy the conditions (1) and therefore there exist vertices D_2 from $X_{\varphi_i}[\varphi_i(a_2)+1,\varphi_i(b_2)]$ and b_3, for which the conditions (2) are fulfilled (after taking $D=D_1, d=b_3$ in (2)). Denote $R_1=X_{\varphi_i}[\varphi_i(b_1)+1,\varphi_i(D_2)-1]$ and $A_2=X_{\varphi_i}[\varphi_i(D_2),\varphi_i(b_2)]$.

Let's continue this procedure. As the graph is finite, then the sets $A_1,A_2,\dots,A_m$ and $R_0,R_1,\dots,R_m$ will be obtained, such that $R_j=X_{\varphi_i}\left[\varphi_i(b_j)+1,\varphi_i(D_{j+1})-1\right]\ at\ m\geq 1$, $A_j=X_{\varphi_i}\left[\varphi_i(D_j),\varphi_i(b_j)\right]\ at$ j $\epsilon\overline{1,m}$, at that $\varphi_i(b_j)-\varphi_i(D_{j-1})\geq K$ (j $\epsilon\overline{1,m+1}$, $D_0=x$), every vertex from A_j is adjacent to b_{j+1} (j $\epsilon\overline{1,m}$), every vertex from R_0 is adjacent to b_1 and there are no vertices in R_0 , the indices of which are above $\varphi_0(b_{m+1})$.

From Lemma 5 we know that every vertex from R_0 is adjacent to every vertex from A_1, every vertex from A_j is adjacent to every vertex from A_{j+1} (j $\epsilon\overline{1,m-1}$) and every vertex from A_m is adjacent to every vertex from R_m.

Let u be an arbitrary vertex from R_j (j $\epsilon\overline{1,m-1}$). We will show, that if u is not adjacent to any vertex from A_j (A_{j+1}), then it is adjacent to all vertices from A_{j+1} (correspondingly: A_j), where j $\epsilon\overline{1,m-1}$). Really, assume the contrary: $u_1\epsilon A_j$, $u_2\epsilon A_{j+1}$ and both are adjacent to u. From $\varphi_i(u)<\varphi_i(u_2)$ we have $\varphi_0(u)<\varphi_0(u_2)$. Then applying Lemma 5 to the triple u_1,u_2,u_3, we will get that u_1 is adjacent to u, which leads to a contradiction. Therefore every vertex $u\epsilon R_j$ (j $\epsilon\overline{1,m-1}$) is adjacent to all vertices of at least one of the sets A_j,A_{j+1}.

Let V_1 be a set consisting of all vertices of R_0 and those vertices from R_1, which are adjacent to all vertices of A_1. Let V_j be a set consisting of all vertices of $R_{j-1}\backslash V_{j-1}$ and those vertices from R_j, which are adjacent to all vertices of A_j (j $\epsilon\overline{1,m-1}$), and $V_m=(R_{m-1}\backslash V_{m-1})\cup R_m$. It is easy to see that the graph G contains as a subgraph a generalized 1-caterpillar satisfying the conditions of Lemma 4. This proves the theorem.

5. Estimation of Algorithm Complexity

From the definition of φ_0 and its second property we have $B_{\varphi_0}(G)\leq\Delta-1$, where Δ is the maximal degree of vertices. On the other hand we have a trivial lower bound:

$$B_{\varphi_0}(G)\geq B(G)\geq\left\lceil\frac{\Delta}{2}\right\rceil. \text{ Therefore } \left\lceil\frac{\Delta}{2}\right\rceil\leq B(G)\leq\Delta-1.$$

First we will estimate the running time of the step i. The vertex z can be found checking $|M|$ vertices and $|M|$ elementary operations are sufficient for the swap of sets $(M,\{z\})$ (for the assignment of numbers). The rest part of the step i, i.e., the problem of finding vertices a_j as well as realization of swaps $(\{a_j\},S_j)$ is equivalent to one pass of known bubble algorithm, and

therefore will require no more than $|M|$ comparisons of indices and $|M|$ operations for the reassignments of numbers. Therefore the step i will require O($|M|$) elementary operations.
As the vertex set M forms a clique (the corresponding intervals contain the interval $\hat{z}$), then $|M| \leq K+1$. Besides the length of the edge (x, y) no more than $2K$, because $\left\lfloor\frac{\Delta}{2}\right\rfloor \leq K \leq \Delta-1$. So, in order to decrease the length of the edge (x, y) to K no more than K steps are needed. Therefore, to achieve a situation where y is not adjacent to an edge with length over K, $O(K^2)$ elementary operations are sufficient. Finally, due to the fact that $\left\lfloor\frac{\Delta}{2}\right\rfloor \leq K \leq \Delta-1$, the bandwidth minimization problem for an interval graph with number of vertices n and with maximal vertex degree Δ, can be solved using $O(\Delta^2 n \log_2 \Delta)$ elementary operations.

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Պոլինոմիալ բարդությամբ ալգորիթմ ինտերվալ գրաֆների բարձրությունը գտնելու համար

Դ. Մուրադյան

Ամփոփում

Դիցուք G-ն կապակցված գրաֆ է X գագաթների և U կողերի բարձրությամբ: Յուրաքանչյուր փոխմիարժեք φ համապատասխանություն, որ գագաթների X բազմությունը արտապատկերում է {1, 2, . . . , |X|} բազմության վրա, կոչվում է G գրաֆի համարակալում: $B_\varphi(G)$ = max $|\varphi(u) - \varphi(v)|$ թիվը, որտեղ մաքսիմումը վերցվում է ըստ գրաֆի բոլոր կողերի, սահմանվում է որպես φ համարակալման բարձրություն: G գրաֆի բարձրությունը սահմանվում է որպես B(G) = min$B_\varphi(G)$, որտեղ մինիմումը վերցվում է ըստ գրաֆի բոլոր համարակալումների: Ինտերվալ գրաֆը սահմանվում է որպես թվային առանցքի վրա տրված ինտերվալների ինչ-որ ընտանիքի հատումների գրաֆ:

Աշխատանքում բերվում է ինտերվալ գրաֆների բարձրությունը գտնող պոլինոմիալ բարդությամբ ալգորիթմ: Ալգորիթմն ունի $O(n\Delta^2\log(\Delta))$ բարդություն, որտեղ n-ը գրաֆի գագաթների քանակն է, իսկ Δ-ն՝ գագաթների մեծագույն աստիճանը:

Алгоритм полиномиальной сложности для нахождения высоты графов интервалов

Д. Мурадян

Аннотация

Пусть G=(X, U) – граф со множеством вершин X и ребер U. Каждое взаимно-однозначное отображение $\varphi: X \to \{1, 2, \ldots, |X|\}$ назовем его нумерацией. При этом число $|\varphi(x) - \varphi(y)|$ назовем длиной ребра (x,y), а числа $B_\varphi(G) = \max_{(x,y)\in U} |\varphi(x) - \varphi(y)|$ и $B(G) = \min_{\varphi} B_\varphi(G)$, где минимум берется по всевозможным нумерациям графа G, соответственно - высотой нумерации φ и графа G. Граф интервалов определяется как граф пересечений семейства интервалов данных на числовой прямой.

В настоящей работе приводится алгоритм полиномиальной сложности для нахождения высоты произвольного графа интервалов. Алгоритм имеет сложность $O(n\Delta^2\log(\Delta))$, где n – количество вершин, а Δ – максимальная степень вершин графа.